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Mathematical Methods in Quantum Mechanics I

2nd Exercise Sheet

Exercise 5:

Prove the following claims:

1. Let $(X, \|\cdot\|)$ be a Banach space, let $A : \mathcal{D}(A) \subseteq X \to X$ be a closed operator and $g : [0,T] \to (\mathcal{D}(A), \|\cdot\|_A)$ be continuous where $\|\cdot\|_A$ is a graph norm. Then

$$A \int_0^T g(t)\mathrm{d}t = \int_0^T Ag(t)\mathrm{d}t\,.$$

   Hint: Integrals can be understood as Riemannian integrals.
2. Let $g : [0,T] \to H^2(\mathbb{R}^3)$ be continuous. Then

$$-\Delta_x \int_0^T g(t)\mathrm{d}t = \int_0^T -\Delta_x g(t)\mathrm{d}t\,.$$

Exercise 6:

Let $\psi \in \mathcal{S}([0,\infty) \times \mathbb{R}^3)$ s.t.

$$i\hbar\partial_t \psi(t,x) = -\frac{\hbar^2}{2m}\Delta\psi(t,x) + V(x)\psi(t,x)\,.$$

Define $\varphi(t,x) := \psi\left(\hbar t, \frac{\hbar}{\sqrt{2m}}x\right)$ and $\tilde{V}(x) := V\left(\left(\frac{\hbar}{\sqrt{2m}}\right)x\right)$. Show that

$$i\partial_t\varphi(t,x) = -\Delta\varphi(t,x) + \tilde{V}\varphi(t,x)\,.$$

Exercise 7:

Show the following: Let X be a Banach space, $K : X \to X$ be linear operator s.t. $\|K\| < 1$. Then $I + K$ is invertible and $(I+K)^{-1} = \sum_{n=0}^{\infty}(-1)^n K^n$.

Exercise 8:

Define $X := (\{\varphi \in L^2(\mathbb{R}) | \varphi \in C^1(\mathbb{R}), \nabla\varphi \in L^2(\mathbb{R})\}, \|\cdot\|_2 + \|\nabla\cdot\|_2)$. Show that

$$f_n = \exp\left[-\left(\frac{1}{n} + |x|^2\right)^{\frac{1}{2}}\right]\,.$$

is a Cauchy sequence in X but f_n does not converge in X. Furthermore show that $f_n \to f$ in $H^1(\mathbb{R})$.