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Mathematical Methods in Quantum Mechanics I

4th Exercise Sheet

Exercise 12:
Let V and T be densely defined linear operators on a Hilbert space $\mathcal{H}$ with $\mathcal{D}(T) \subset \mathcal{D}(V)$. Consider two assumptions:

1. For some $a, b \in \mathbb{R}$ and all $\psi \in \mathcal{D}(T)$
$$\|V\psi\| \leq a\|T\psi\| + b\|\psi\| \,. \tag{1}$$
2. For some $\tilde{a}, \tilde{b} \in \mathbb{R}$ and all $\psi \in \mathcal{D}(T)$
$$\|V\psi\|^2 \leq \tilde{a}^2\|T\psi\|^2 + \tilde{b}^2\|\psi\|^2 \,. \tag{2}$$

Show the following statements:

1. If (2) holds, then (1) holds with $a = \tilde{a}$ and $b = \tilde{b}$.
2. If (1) holds, then (2) holds for $\tilde{a}^2 = (1+\epsilon)a^2$ and $\tilde{b}^2 = (1+\epsilon^{-1})b^2$ for each $\epsilon > 0$. Thus the infimum of all a in (1) is the same as the infimum of all $\tilde{a}$ in (1).

Exercise 13:
Consider a Hamiltonian describing n electron atom in Born-Oppenheimer approximation. Let $\mathcal{H} = H^2(\mathbb{R}^{3n})$,
$$T = \sum_{j=1}^{n} -\Delta_j \;,\; \mathcal{D}(T) = \mathcal{H}$$
where $-\Delta_j$ is a kinetic energy operator of j-th electron and V is a multiplication operator defined using the function
$$v(x) = -\sum_{j=1}^{n} \frac{Z}{|x_j|} + \sum_{k>j=1}^{n} \frac{1}{|x_j - x_k|}$$
where $|x_j|$ is a coordinate of the j-th electron.

Show the following:

1. V is bounded w.r.t. T with $a < 1$, i.e. there exists $0 \leq a < 1$ and $b \in \mathbb{R}_0^+$ s.t. (1) holds
2. $T + V$ is lower semibounded, i.e. there exists $a \in \mathbb{R}$ s.t. $\forall \varphi \in \mathcal{H}$: $\langle \varphi, (T+V)\varphi \rangle \geq a\|\varphi\|^2$,
3. $T + V$ is self-adjoint.

Exercise 14:
Let $H_0^1(\mathbb{R}^+) = \overline{C_c^\infty(\mathbb{R}^+)}$. Then we define "momentum operator" $P : H_0^1(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ as
$$Pf(x) = -i\frac{\mathrm{d}}{\mathrm{d}x}f(x) \,.$$

Show the following:

1. P is symmetric and P^* is not symmetric,
2. P is closed and
3. P is not essentially self-adjoint.