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Mathematical Methods in Quantum Mechanics I

5th Exercise Sheet

Exercise 15:
Let $U(t) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), U(t)\psi(x) = \psi(R_t x)$ where

$$R_t x = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \cos t - x_2 \sin t \\ x_1 \sin t + x_2 \cos t \end{pmatrix} .$$

Show the following statements:

1. $U(t)$ is strongly continuous group,
2. $U(t)$ has a generator in the form of multiple of angular momentum operator L,
3. angular momentum operator $L = x_1 p_2 - x_2 p_1$ is self-adjoint on an appropriate domain.

Exercise 16:
Let H be a Hilbert space and $A, B \in \mathcal{L}(H)$. Prove the Trotter product formula

$$e^{A+B} = \lim_{N\to\infty} \left(e^{A/N} e^{B/N}\right)^N .$$

Hint: Define $C_N := e^{(A+B)/N}$ and $D_N := e^{A/N} e^{B/N}$ to show that $\|C_N - D_N\| = \mathcal{O}(N^{-2})$ and compare telescopic sums C_N^N and D_N^N.

Exercise 17:
Let $\psi \in H^1(\mathbb{R}^3)$. Prove that

$$\int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|} \mathrm{d}x \leq \|\nabla\psi\|_2 \|\psi\|_2 .$$

Equality holds only if

$$\psi(x) \sim e^{-c|x|}$$

for some constant $c > 0$.
Hint: Use that $\frac{1}{|x|} = \frac{1}{2}\nabla\left(\frac{x}{|x|}\right)$.