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WS2019/20
9.12.2019

Mathematical Methods in Quantum Mechanics I

8th Exercise Sheet

Exercise 25:
Let $\mathcal{H}$ be a Hilbert space and A be a bounded self-adjoint operator.

1. Prove that $\|A\| = \sup_{\|\psi\|=1} |\langle \psi, A\psi \rangle|$.
2. Find an example of operator for which the equality fails.
   Hint: Look for nonself-adjoint operator.

Exercise 26:
Let $\mathcal{H}$ be a Hilbert space and B be a self-adjoint operator. Furthermore there exists $c > 0$ s.t. $\|B\psi\| \geq c\|\psi\|$ for all $\psi \in D(B)$. Prove the following

1. $\mathrm{Ran}(B)$ is closed.
2. $\mathrm{Ran}(B)$ is dense.
3. $\|B^{-1}\varphi\| \leq \frac{1}{c}\|\varphi\|$ holds for all $\varphi \in \mathcal{H}$.

Exercise 27:
Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{L}(X)$. Prove that $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ exists and is equal to $\inf_n \|T^n\|^{\frac{1}{n}}$ as follows:

1. Set $a_n = \log \|T^n\|$ and prove that $a_{m+n} \leq a_m + a_n$.
2. For a fixed positive integer m set $n = mq + r$ where q and r are positive integers for $0 \leq r \leq m - 1$. Using 1. conclude that

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_m}{m} .$$

3. Prove that $\lim_{n\to\infty} \frac{a_n}{n} \leq \frac{a_m}{m}$ and thus the desired equality.

Exercise 28:
Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ s.t. $TT^* = T^*T$ i.e. T is normal. Then

$$r(T) = \|T\| .$$