Institut für Analysis
Dr. Ioannis Anapolitanos
Dr. Michal Jex

WS2019/20
16.12.2019

Mathematical Methods in Quantum Mechanics I

9th Exercise Sheet

Exercise 29:
Let $\mathcal{H}$ be a Hilbert space.

1. Let A be a self-adjoint operator. Prove that for each $\lambda \in \rho(A)$ we have
$$\|(\lambda - A)^{-1}\| = \frac{1}{d(\lambda, \sigma(A))} \tag{1}$$
2. Find an example of operator for which the equality (1) fails.
Hint: Look for non-normal operator.

Exercise 30:
Let $\psi \in H^1(\mathbb{R}^3)$. Show that
$$\lim_{h\to\infty} \left\| \frac{\psi(x-h)}{x} \right\| = 0\,.$$

Exercise 31:
Let $f \in \{\psi \in C(\mathbb{R}^d) | \lim_{|x|\to\infty} \psi = 0\}$ and $M_f : H^1(\mathbb{R}^d) \to L^2(\mathbb{R}^d), M_f\varphi := f\varphi$. Show that M_f is compact.

Exercise 32:
Let $\mathcal{H}$ be a Hilbert space and $S \in \mathcal{L}(\mathcal{H})$ self-adjoint. Furthermore define numerical range $W(S) := \{(x, Sx) | x \in \mathcal{H}, \|x\| = 1\}$. Show that $W(S) \subseteq \mathbb{R}$ and
$$\sigma(S) \subseteq [m, M] \subseteq [-\|S\|, \|S\|]$$
where $m := \inf\{(Sx|x) | \, \|x\| = 1\}$ and $M := \sup\{(Sx|x) | \, \|x\| = 1\}$. Moreover show that $m, M \in \sigma(S)$ and $M = \|S\|$ or $m = -\|S\|$.