Institut für Analysis
Dr. Ioannis Anapolitanos
Dr. Michal Jex

WS2019/20
13.1.2020

Mathematical Methods in Quantum Mechanics I

10th Exercise Sheet

Exercise 33:
Let $\lambda > 0$ then the operator $U_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is defined as

$$[U_\lambda \phi](x) := \lambda^{n/2}\phi(\lambda x)\,.$$

Show that:

1. U_λ is unitary, $U_\lambda^{-1} = U_{\lambda^{-1}}$ and U_λ leaves $H^s(\mathbb{R}^n)$ invariant,
2. On $\mathcal{S}(\mathbb{R}^n)$ the following holds

$$U_\lambda \Delta U_\lambda^{-1} = \frac{1}{\lambda^2}\Delta\,, \quad U_\lambda V(x) U_\lambda^{-1} = V(\lambda x)\,,$$

with V being multiplication operator satisfying $V\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$.

Exercise 34:
Let $k \geq -\frac{n}{2}$ and $k \neq -2$. Consider operators $H_0 = -A\Delta + Br^k$ for $A, B \in \mathbb{R}^+$ and $H_1 = -\Delta + r^k$ which are essentially self-adjoint on $\mathcal{S}(\mathbb{R}^n)$. Let the operator H_0 have an eigenfunction ψ_0, $\|\psi_0\| = 1$ with the eigenvalue E_0, i.e.

$$H_0\psi_0 = E_0\psi_0\,.$$

Show that the operator H_1 has an eigenfunction ψ_1 with an eigenvalue E_1. Furthermore prove the relations $\psi_1 = U_\lambda \psi_0$ and $E_1 = \frac{\lambda^2}{A}E_0$ where $\lambda = \left(\frac{A}{B}\right)^{\frac{1}{k+2}}$.
Hint: Use the solution of previous Problem.

Exercise 35:
Let E_M, E_{M^+} and E_{M^-} be an infimum of the energy for an atom M, cation M^+ or anion M^- respectively. Prove that

1. $E_H + E_H < E_{H^+} + E_{H^-}$,
2. if $E_H + E_M < E_{H^+} + E_{M^-}$ fails then M^- has a ground state.