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Mathematical Methods in Quantum Mechanics I

11th Exercise Sheet

Exercise 36: *Newton's Shell Theorem*
Prove that the following equality holds

$$\int_{|x|=r} \frac{d\sigma(x)}{|x-R|} = \frac{4\pi r^2}{\max\{r, |R|\}}$$

where $d\sigma(x)$ is a surface measure on a sphere and $R \in \mathbb{R}^3$.

Exercise 37:
Let H_{He} be a Schrödinger operator describing Helium atom in Born-Oppenheimer approximation. Show that the ground state energy E_{He} is smaller or equal to $-\frac{11}{8}$.
Hint: Use the test function in the form of a product of two hydrogen type functions.

Exercise 38:
Let H be an operator bounded from below and satisfying IMS localization formula. Furthermore let $E \in \mathbb{R}$ be below the ionization threshold of H. Prove the following

1. Let $e^{a|x|}(H-E)\psi \in L^2(\mathbb{R}^n)$ for some $a > 0$ then there exists $\delta > 0$ s.t. $e^{\delta|x|}\psi(x) \in L^2(\mathbb{R}^n)$.
2. Let $e^{b|x|}\varphi \in L^2(\mathbb{R}^n)$ for some $b > 0$. If $\lambda \in \mathbb{R} \cap \rho(H)$ is below the ionization threshold of H then there exists $\delta > 0$ s.t. $e^{\delta|x|}(H-\lambda)^{-1}\varphi(x) \in L^2(\mathbb{R}^n)$.

Exercise 39: *Hardy's inequality for 1D*
Show that for every function $\psi \in H_0^1(\mathbb{R}^+)$ the following holds

$$\int_{\mathbb{R}^+} |\psi'(x)|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^+} \frac{|\psi(x)|^2}{x^2} dx \,. \tag{1}$$

Furthermore show that the inequality (1) is non-attainable and optimal, i.e.

1. *non-attainability* There is no non-trivial function such that, the equality holds in (1),
2. *optimality* We can not make the constant 1/4 on the right hand side bigger.