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# Mathematical Methods in Quantum Mechanics I

## 12th Exercise Sheet

**Exercise 40:**
Let $A$ be a self-adjoint operator on $\mathcal{H}$. Let $P_\Omega := \chi_\Omega(A)$ where $\chi_\Omega$ is a characteristic function of the measurable set $\Omega \subset \mathbb{R}$. Show that the family of operators $\{P_\Omega\}$ has the following properties:

1. Each $P_\Omega$ is an orthogonal projection.
2. $P_\emptyset = 0$, $P_\mathbb{R} = I$.
3. Let $N \in \mathbb{N} \cup \{0\}$. If $\Omega = \bigcup_{n=1}^N \Omega_n$ with $\Omega_n \cap \Omega_m = \emptyset$ if $n \neq m$, then $P_\Omega \psi = \sum_{n=1}^N P_{\Omega_n} \psi$, $\forall \psi \in \mathcal{H}$.
4. $P_{\Omega_1} P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$.

**Exercise 41:**
Let $A$ be a self-adjoint operator and define $U(t) = e^{itA}$. Using Spectral Theorem for self-adjoint operators, prove

1. For each $t \in \mathbb{R}$, $U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all $s, t \in \mathbb{R}$.
2. If $\varphi \in \mathcal{H}$ and $t \to t_0$, then $U(t)\varphi \to U(t_0)\varphi$.
3. For $\psi \in D(A)$, $\frac{U(t)\psi - \psi}{t} \to iA\psi$ as $t \to 0$.
4. If $\lim_{t\to 0} \frac{U(t)\psi - \psi}{t} = iA\psi$ exists, then $\psi \in D(A)$.

**Exercise 42:**
Let $A$ be a self-adjoint operator and let $P_\Omega := \chi_\Omega(A)$. Prove the following

1. $\lambda \in \sigma(A)$ if and only if for all $\epsilon > 0 : P_{(\lambda-\epsilon,\lambda+\epsilon)} \neq 0$ .
2. $\lambda \in \sigma_d(A)$ if and only if $\lambda \in \sigma(A)$ and there exists $\epsilon > 0$ s.t. $P_{(\lambda-\epsilon,\lambda+\epsilon)}\mathcal{H} < \infty$.
3. $\lambda \in \sigma_{\text{ess}}(A)$ if and only if $\dim(P_{(\lambda-\epsilon,\lambda+\epsilon)}\mathcal{H}) = \infty$ for all $\epsilon > 0$.

**Exercise 43:**
Let $A : \mathcal{H} \to \mathcal{H}$ be bounded self-adjoint positive operator namely $\langle \varphi, A\varphi \rangle \geq 0$, $\forall \varphi \in \mathcal{H}$. Let $P : \mathcal{H} \to \mathcal{H}$ be an orthogonal projection with $\dim(P\mathcal{H}) = n$. Show that

1. $\sigma_{ess}(A - P) \subseteq [0, \infty)$.
2. $A - P$ has at most $n$ negative eigenvalues.