Retour to compact operators

Thm 6.8 Let X, Y be Hilbert spaces and $K_n : X \to Y$ a sequence of compact operators. If $K_n \to K$ in the operator norm for some $K : X \to Y$ then K is compact.

Sketch of pf Let $(x_m)_{m \in \mathbb{N}} \subset X$ bounded.

K_1 compact $\rightsquigarrow$ $\exists (x_m)_{m \in A_1}$ $A_1 \subset \mathbb{N}$ so that $(K_1 x_m)_{m \in A_1}$ is convergent.

But K_2 compact, $(x_m)_{m \in A_1}$ bounded $\Rightarrow$ $\exists (x_m)_{m \in A_2}$, $A_2 \subset A_1$ so that $(K_2 x_m)_{m \in A_2}$ is convergent. Continuing iteratively we find a family of subsequences $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$ such that $(K_n x_m)_{m \in A_n}$ is convergent. Choose

$x_{m_1} \in A_1$, $x_{m_2} \in A_2$, $\ldots$, $x_{m_k} \in A_k$.

with m_k increasing then

$(K_n x_{m_\ell})_{\ell \in \mathbb{N}}$ is convergent $\forall n \in \mathbb{N}$.

Let $y_n = \lim_{l\to\infty} K_n x_{m_l}$. Then y_n is convergent and $y = \lim_{n\to\infty} y_n$ turns out to be the limit of $K x_{m_l}$.

<u>Thm 6.9</u> Let $f \in C_c^1(\mathbb{R}^n)$. Then the operator $T_f : H^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

$T_f \psi = f\psi$ is compact.

<u>Idea of pf</u> $T_f = J \circ G_f$

where $G_f : H^1(\mathbb{R}^n) \to B := \{g \in H^1(\mathbb{R}^n) : \operatorname{supp} g \subset \operatorname{supp} f\}$

$G_f \psi = f\psi$ $J : B \subset L^2(\mathbb{R}^n)$, $Jg = g$.

$H_0^1(U) = \overline{C_c^\infty(U)}^{H^1}$, So

continuous by Leibnitz rule

$$\begin{tikzcd} H^1(\mathbb{R}^n) \arrow[r, "G_f"] \arrow[dr, "T_f"'] & B \arrow[d, "J"] \\ & L^2(\mathbb{R}^n) \end{tikzcd}$$

compact by Rellich Kondrachov

compact as a composition of a compact and a continuous operator.

Thm 6.10 Let $f \in C(\mathbb{R}^n)$ with $\lim_{|x| \to \infty} f(x) = 0$.

then $T_f : H^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ $T_f \psi = f \psi$

is compact. (Exercise session).

Hint combine Thms 6.8, 6.9.

Ex 8.1 Let $H: H^2(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$.

$H\psi = \left(-\Delta - \frac{1}{|x|}\right)\psi$. Then $\sigma_{ess}(H) = [0,\infty)$.
In particular, the ground state energy of H (which is $-\frac{1}{4}$ by Thm 5.5 is in the discrete spectrum of H.

Pf Let $\lambda \in \sigma_{ess}(H)$. Then by the Weyl's criterion $\exists (\varphi_n)_{n\in\mathbb{N}} \subset H^2(\mathbb{R}^3)$ with $\|\varphi_n\|_{L^2} = 1$, $\left\|\left(-\Delta - \frac{1}{|x|} - \lambda\right)\varphi_n\right\| \to 0$ and $\varphi_n \rightharpoonup 0$.

It follows that $\langle \varphi_n, \left(-\Delta - \frac{1}{|x|} - \lambda\right)\varphi_n \rangle \to 0$.

$$\Rightarrow \int |\nabla\varphi_n|^2 - \int \frac{|\varphi_n|^2}{|x|} \to \lambda.$$

But $\int \frac{|\varphi_n|^2}{|x|}dx \le \int |\varphi_n| \frac{|\varphi_n|}{|x|}dx \le \left(\int |\varphi_n|^2 dx\right)^{\frac{1}{2}} \left(\int \frac{|\varphi_n|^2}{|x|^2} dx\right)^{\frac{1}{2}} \overset{\text{Hardy}}{\le} 2\|\nabla\varphi_n\|$

Thus $\int |\nabla\varphi_n|^2 - \int \frac{|\varphi_n|^2}{|x|}dx \ge \|\nabla\varphi_n\|^2 - 2\|\nabla\varphi_n\| \Longrightarrow$

$$\lambda \leftarrow \int |\nabla\varphi_n|^2 - \int \frac{|\varphi_n|^2}{|x|}dx \ge (\|\nabla\varphi_n\|^2 - 1)^2 - 1.$$

Thus φ_n is bounded in the H^1 norm ($\exists\ c>0$ with $\|\varphi_n\|_{H^1} \le c \quad \forall n \in \mathbb{N}$).

Now we show that $\int \frac{|\varphi_n|^2}{|x|} \xrightarrow{n\to\infty} 0.$

Let $\varepsilon > 0$. Then $\exists\ R>0$ such that $\frac{1}{R} < \frac{\varepsilon}{2}$.

Thus $$\int \frac{|\varphi_n|^2}{|x|} dx = \int_{|x|\le R} \frac{|\varphi_n|^2}{|x|} dx + \int_{|x|\ge R} \frac{|\varphi_n|^2}{|x|}.$$

$$\le \int_{|x|\le R} |\varphi_n| \frac{|\varphi_n|}{|x|} dx + \frac{1}{R} \int_{|x|\ge R} |\varphi_n|^2 dx \le \underbrace{\left(\int_{|x|\le R} |\varphi_n|^2 dx\right)^{\frac{1}{2}}}_{\downarrow\ 0} \underbrace{\left(\int \frac{|\varphi_n|^2}{|x|^2} dx\right)^{\frac{1}{2}}}_{\le 2\|\nabla\varphi_n\|\text{ bounded}} + \frac{1}{R}.$$

~~Exercise session~~ To be uploaded

Thus $\int \frac{|\varphi_n|^2}{|x|} dx \to 0$. Therefore,

$$\int |\nabla \varphi_n|^2 dx \to \lambda \quad \text{thus} \quad \lambda > 0.$$

Thus $\sigma_{ess}(H) \subset [0,\infty)$.

Assume now that $\lambda \in [0,\infty)$. Then by Thm 2.5 $\lambda \in \sigma(-\Delta)$. Thus by Thm 7.5 $\forall \varepsilon > 0\ \exists (\varphi_n)_{n\in\mathbb{N}} \subset H^2(\mathbb{R}^3)$

by Ex...

with $\|(\Delta-\lambda)\psi_\varepsilon\| < \varepsilon$.

it follows that $\|(\Delta-\lambda-\frac{1}{|x|})\psi_{\varepsilon,h}\| \le \|(\Delta-\lambda)\psi_{\varepsilon,h}\| + \|\frac{1}{|x|}\psi_{\varepsilon,h}\|$.

where $\psi_{\varepsilon,h}(x) = \psi_\varepsilon(x-h)$. But then.

$$\lim_{|h|\to\infty} \left\|\left(-\Delta-\lambda-\frac{1}{|x|}\right)\psi_{\varepsilon,h}\right\| = \underbrace{\lim_{|h|\to\infty} \|(\Delta-\lambda)\psi_{\varepsilon,h}\|}_{h \text{ independent}} + \underbrace{\lim_{|h|\to\infty} \left\|\frac{\psi_{\varepsilon,h}}{|x|}\right\|}_{\text{Exercise session}}$$

<u>Thm 8.3</u> Assume that $V \in C(\mathbb{R}^n)$ with $\lim_{|x|\to\infty} V(x) = 0$. Then the operator

$$H = H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \quad H = -\Delta + V$$

is self-adjoint and $\sigma_{ess}(H) = [0,\infty)$.

<u>Pf</u> Since $\|V\|_{L^\infty} < \infty$ V is a bounded operator, thus by the Kato-Rellich Theorem $\xrightarrow{\text{Thm 3.8}}$ H is self adjoint.

That $[0,\infty) \subset \sigma_{ess}(H)$ can be shown similarly as in Ex 8.1.

Assume that $\lambda \in \sigma_{ess}(H)$. Then $\exists (\varphi_n)_{n\in\mathbb{N}} \subset H^2$ with $\|(-\Delta+V-\lambda)\varphi_n\| \to 0$, $\|\varphi_n\|=1$ and $\varphi_n \rightharpoonup 0$.

Thus $\int |\nabla\varphi_n|^2 dx + \int V|\varphi_n|^2 dx \to \lambda$. Thus

$(\varphi_n)_{n\in\mathbb{N}}$ is bounded in H^1 and $\varphi_n \rightharpoonup 0$ in L^2.

But by Thm 6.10 $T_V : H^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $T_V\varphi = V\varphi$ is compact. Thus, by Thm. 6.7 $V\varphi_n \xrightarrow{L^2} 0$. $\to$ details to be uploaded.

and thus $\|(-\Delta-\lambda)\varphi_n\| \to 0$. Therefore $\lambda \in \sigma(-\Delta) = [0,\infty)$.

<u>Thm 8.4</u> Assume that $V \in C(\mathbb{R}^n)$ with $\lim_{|x|\to\infty} V(x) = \infty$. Then the operator

$H = -\Delta + V$ with $D(H) = \{\varphi \in L^2(\mathbb{R}^n) : (-\Delta+V)\varphi \in L^2(\mathbb{R}^n)\}$

is self-adjoint and $\sigma_{ess}(H) = \emptyset$.

<u>Pf</u> We omit (at least for now) the proof that H is self-adjoint and that $H^2 \subset D(H)$.