

Retour to compact operators

Thm 6.8 Let X, Y be Hilbert spaces.

and $K_n: X \rightarrow Y$ a sequence of compact operators. If $K_n \rightarrow K$ in the operator norm for some $K: X \rightarrow Y$ then K is compact.

Sketch of pf Let $(x_m)_{m \in \mathbb{N}} \subset X$ bounded.

K_1 compact $\leadsto \exists (x_m)_{m \in A_1}, A_1 \subset \mathbb{N}$ so that $(K_1 x_m)_{m \in A_1}$ is convergent.

But K_2 compact, $(x_m)_{m \in A_1}$ bounded \Rightarrow

$\exists (x_m)_{m \in A_2}, A_2 \subset A_1$ so that $(K_2 x_m)_{m \in A_2}$ is convergent. Continuing iteratively we find a family of subsequences.

$A_1 \supset A_2 \supset \dots \supset A_n \supset \dots$ such that

$(K_n x_m)_{m \in A_n}$ is convergent. Choose

$x_{m_1} \in A_1, x_{m_2} \in A_2, \dots, x_{m_k} \in A_k$.

with m_k increasing then.

$(K_n x_{m_k})_{k \in \mathbb{N}}$ is convergent $\forall n \in \mathbb{N}$.

Let $y_n = \lim_{l \rightarrow \infty} K_n x_{ml}$. Then y_n is convergent and $y = \lim_{n \rightarrow \infty} y_n$ turns out to be the limit of $K x_{ml}$.

Thm 6.9 Let $f \in C_c^1(\mathbb{R}^n)$. Then the operator $T_f: H^1(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$.

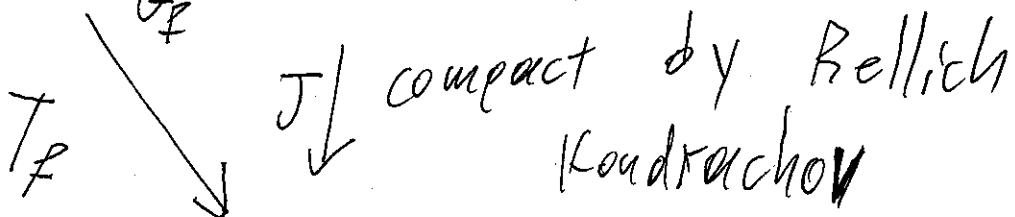
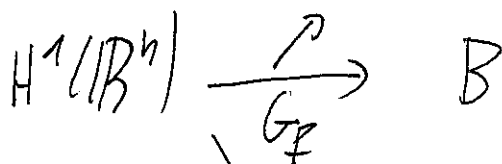
$T_f \psi = f \psi$ is compact.

Idea of proof $T_f = J \circ G_f$

where $G_f: H^1(\mathbb{R}^n) \rightarrow B := \{g \in H^1(\mathbb{R}^n) : \text{supp } g \subset \text{supp } f\}$

$G_f \psi = f \psi$ $J: B \subset L^2(\mathbb{R}^n)$, $Jg = g$.

$H_0^1(U) = \overline{C_c^\infty(U)}^{H^1}$. So continuous by Leibnitz rule



$L^2(\mathbb{R}^n)$

compact as a composition of a compact and a continuous operator.

Thm 6.10 Let $f \in C(\mathbb{R}^n)$ with $\lim_{|x| \rightarrow \infty} f(x) = 0$

then $T_f: H^1(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ $T_f \psi = f\psi$

is compact (Exercise session).

Hint combine Thms 6.8, 6.9.

Ex 8.1 Let $H: H^2(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$

$H\psi = (-\Delta - \frac{1}{|x|})\psi$. Then $\sigma_{\text{ess}}(H) = [0, \infty)$.

In particular, the ground state energy of H (which is $-\frac{1}{4}$ by Thm 5.5) is in the discrete spectrum of H .

PP Let $\lambda \in \sigma_{\text{ess}}(H)$. Then by the

Weyl's criterion $\exists (\psi_n)_{n \in \mathbb{N}} \subset H^2(\mathbb{R}^3)$

with $\|\psi_n\|_{L^2} = 1$, $\|(-\Delta - \frac{1}{|x|} - \lambda)\psi_n\| \rightarrow 0$

and $\psi_n \rightarrow 0$.

It follows that $\langle \psi_n, (-\Delta - \frac{1}{|x|} - \lambda)\psi_n \rangle \rightarrow 0$.

$$\Rightarrow \int |\nabla \psi_n|^2 - \int \frac{|\psi_n|^2}{|x|} \rightarrow \lambda.$$

But $\int \frac{|\psi_n|^2}{|x|} dx \leq \int |\psi_n| \frac{|\psi_n|}{|x|} dx \leq \left(\int |\psi_n|^2 dx\right)^{\frac{1}{2}} \left(\int \frac{|\psi_n|^2}{|x|^2} dx\right)^{\frac{1}{2}} \stackrel{\text{Hardy}}{\leq} 2\|\nabla \psi_n\|$

Thus $\int |\nabla \psi_n|^2 - \int \frac{|\psi_n|^2}{|x|} dx \geq \|\nabla \psi_n\|^2 - 2\|\nabla \psi_n\| \Rightarrow$

$$\lambda \leftarrow \int |\nabla \psi_n|^2 - \int \frac{|\psi_n|^2}{|x|} dx \geq (\|\nabla \psi_n\|^2 - 1)^2 - 1.$$

Thus ψ_n is bounded in the H^1 norm
 ($\exists c > 0$ with $\|\psi_n\|_{H^1} \leq c \quad \forall n \in \mathbb{N}$).

Now we show that $\int \frac{|\psi_n|^2}{|x|} dx \xrightarrow{n \rightarrow \infty} 0$.

Let $\varepsilon > 0$. Then $\exists R > 0$ such that $\frac{1}{R} < \frac{\varepsilon}{2}$.

$$\text{Thus } \int \frac{|\psi_n|^2}{|x|} dx = \int_{|x| \leq R} \frac{|\psi_n|^2}{|x|} dx + \int_{|x| \geq R} \frac{|\psi_n|^2}{|x|} dx.$$

$$\leq \int_{|x| \leq R} |\psi_n| \frac{|\psi_n|}{|x|} dx + \frac{1}{R} \int_{|x| \geq R} |\psi_n|^2 dx \leq \left(\int_{|x| \leq R} |\psi_n|^2 dx \right)^{\frac{1}{2}} \left(\int_{|x| \leq R} \frac{|\psi_n|^2}{|x|^2} dx \right)^{\frac{1}{2}} + \frac{1}{R} \int_{|x| \geq R} |\psi_n|^2 dx$$

$\xrightarrow{\text{To be uploaded } 0}$
 $\leq \|\psi_n\| \text{ bounded}$

Thus $\int \frac{|\psi_n|^2}{|x|} dx \rightarrow 0$. Therefore,

$$\int |\nabla \psi_n|^2 dx \rightarrow \lambda \quad \text{thus } \lambda > 0.$$

Thus $\sigma_{\text{ess}}(H) \subset [0, \infty)$.

Assume now that $\lambda \in [0, \infty)$. Then by

Thm 2.5 $\lambda \in \sigma(-\Delta)$. Thus by Thm 7.5 $\exists (\psi_n)_{n \in \mathbb{N}} \subset H^2(\mathbb{R}^3)$
 by Ex...

with $\|(\Delta - \lambda)\psi_\varepsilon\| < \varepsilon$.

It follows that $\|(\Delta - \lambda)\psi_{\varepsilon, h}\| \leq \|(\Delta - \lambda)\psi_\varepsilon\| + \left\| \frac{1}{|x|} \psi_{\varepsilon, h} \right\|$.

where $\psi_{\varepsilon, h}^{(x)} = \psi_\varepsilon(x-h)$. But then

$$\lim_{|h| \rightarrow \infty} \left\| \left(-\Delta - \lambda - \frac{1}{|x|} \right) \psi_{\varepsilon, h} \right\| = \lim_{|h| \rightarrow \infty} \|(\Delta - \lambda)\psi_{\varepsilon, h}\| + \lim_{|h| \rightarrow \infty} \left\| \frac{\psi_{\varepsilon, h}}{|x|} \right\|$$

h independent

Exercise session

Thm 8.3 Assume that $V \in C(\mathbb{R}^n)$ with $\lim_{|x| \rightarrow \infty} V(x) = 0$. Then the operator $H = H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$, $H = -\Delta + V$ is self-adjoint and $\sigma_{\text{ess}}(H) = [0, \infty)$.

PF Since $\|V\|_{L^\infty} < \infty$ V is a bounded operator, thus by the Kato-Bellis Theorem ^{Thm 3.8} H is self-adjoint.

That $[0, \infty) \subset \sigma_{\text{ess}}(H)$ can be shown similarly as in Ex 8.1.

Assume that $\lambda \in \sigma_{\text{ess}}(H)$. Then $\exists (\varphi_n)_{n \in \mathbb{N}} \subset H^2$
 with $\|(-\Delta + V - \lambda)\varphi_n\| \rightarrow 0$, $\|\varphi_n\| = 1$ and $\varphi_n \rightarrow 0$.

Thus $\int |\nabla \varphi_n|^2 dx + \int V |\varphi_n|^2 dx \rightarrow \lambda$. Thus

$(\varphi_n)_{n \in \mathbb{N}}$ is bounded in H^1 and $\varphi_n \rightarrow 0$
 in L^2 .

But by Thm 6.10 $T_V: H^1(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$

$T_V \varphi = V\varphi$ is compact. Thus, by

Thm. 6.7 $V\varphi_n \xrightarrow{L^2} 0$ \rightarrow details to be uploaded.

and thus $\|(-\Delta - \lambda)\varphi_n\| \rightarrow 0$. Therefore

$$\lambda \in \sigma(-\Delta) = [0, \infty).$$

Thm 8.4 Assume that $V \in C(\mathbb{R}^n)$
 with $\lim_{|x| \rightarrow \infty} V(x) = \infty$. Then the operator

$H = -\Delta + V$ with $D(H) = \{\varphi \in L^2(\mathbb{R}^n) : (-\Delta + V)\varphi \in L^2(\mathbb{R}^n)\}$
 is self-adjoint and $\sigma_{\text{ess}}(H) = \emptyset$.

PF We omit (at least for now) the
 proof that H is self-adjoint and
 that $H^2 \subset D(H)$.