Let X be a vector space with a norm $\|\cdot\|$ [rsp. inner product $\langle\cdot,\cdot\rangle$]. X is called a <u>Banach space</u> [rsp. Hilbert space] if every sequence in X is convergent. If in addition X is a complex vector space it is called a <u>complex Banach space</u> [rsp. complex Hilbert space].

<u>Example 0.1</u>/ The $L^2(\mathbb{R}^3) = \{ \psi : \mathbb{R}^3 \to \mathbb{C},\ \int_{\mathbb{R}^3} |\psi|^2 dx < \infty \}$ equipped with the inner product

$\langle \psi, \varphi \rangle = \int_{\mathbb{R}} \overline{\psi(x)}\, \varphi(x)\, dx$ is a complex Hilbert space and therefore a complex Banach space. (see e.g. Real Analysis Gerald Folland Chapter 6.1).

<u>Example 0.2</u> Let $X = \{ \varphi \in L^2(\mathbb{R}) : \varphi \in C^1(\mathbb{R}),\ \varphi' \in L^2(\mathbb{R}) \}$ equipped with the norm

$\|\varphi\|_X = \|\varphi\|_{L^2(\mathbb{R})} + \|\varphi'\|_{L^2(\mathbb{R})}$.

Then X is not a Banach space. (Exercise session). Hint: let

$u_n = \exp\left(-\left(\frac{1}{n}+|x|^2\right)^{\frac{1}{2}}\right)$. Then u_n is a Cauchy sequence in X and since

$u_n' \xrightarrow{n\to\infty} \exp(-|x|)$ in $L^2(\mathbb{R})$ but

$\exp(-|x|) \notin X$ (because it is not differentiable) u_n is not convergent in X (because if $u_n \to u$ in X then $u_n \to u$ in $L^2(\mathbb{R})$ so $u(x) = \exp(-|x|) \in X$, contradiction).

Since completeness (every Cauchy sequence is convergent) is an important property and Δ is important in quantum mechanics we would like to extend X to a Banach space. To do so we need to extend the notion of differentiability.

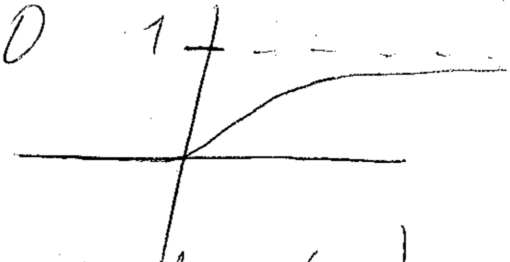
1 Weak derivatives and Sobolev spaces.

Notation $C_c^\infty(\mathbb{R}^n) = \{\phi : \mathbb{R}^n \to \mathbb{C} : \phi \in C^\infty, \operatorname{supp}\phi \text{ compact}\}$

where $\operatorname{supp}\phi := \overline{\{x \in \mathbb{R}^n : \phi(x) \neq 0\}}$

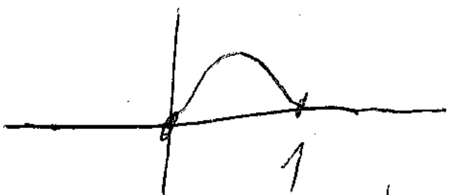
A $\phi \in C_c^\infty(\mathbb{R}^n)$ is called a test function.

<u>Ex 1.1</u> $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$



Then $f \in C^\infty(\mathbb{R})$ (proof by induction).

$g(x) = f(x) f(1-x)$ is in $C_c^\infty(\mathbb{R})$.



Multiplication of a $C_c^\infty(\mathbb{R})$ with a $C^\infty(\mathbb{R})$ function gives a $C_c^\infty(\mathbb{R})$ function etc.

$L^1_{loc}(\mathbb{R}^n) = \{\phi : \mathbb{R}^n \to \mathbb{C} : \phi \in L^1(K), \forall K \subseteq \mathbb{R}^n$ with K compact$\}$.

<u>Ex 1.2</u> (i) $\phi(x) = e^{x^2}$, $\phi \in L^1_{loc}(\mathbb{R})$ because ϕ is bounded in K for all compact K and therefore in $L^1(K)$.

(ii) $f(x) = \frac{1}{|x|}$, $f \notin L^1_{loc}(\mathbb{R})$, because $f \notin L^1([-1,1])$.

$\alpha = (\alpha_1, \dots, \alpha_n)$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$, $|\alpha| = |\alpha_1| + \dots + |\alpha_n|$

Let $u \in C_c^\infty(\mathbb{R}^n)$. Then we have that $\int_{\mathbb{R}^n} u \, \partial^\alpha \phi \, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \partial^\alpha u \, \phi \, dx$.

and in fact if $w, u \in C_c^\infty(\mathbb{R}^n)$ then

$$w = \partial^\alpha u \iff \left(\int_{\mathbb{R}^n} u \, \partial^\alpha \phi \, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} w \phi \, dx, \ \forall \phi \in C_c^\infty(\mathbb{R}^n) \right) \quad (1)$$

The notion of weak derivative maintains this property

<u>Def 1.1</u> Let $u, w \in L^1_{loc}(\mathbb{R}^n)$. We say that $w = \partial^\alpha u$ (α^{th} weak partial derivative of u) if (1) holds.

<u>Prop 1.2</u> If $u \in L^1_{loc}(\mathbb{R}^n)$ has a weak derivative then it is unique.

<u>Example 1.3</u> Let $u : \mathbb{R} \to \mathbb{R}$ $u(x) = e^{-|x|}$

Then in the sence of weak derivatives.

$u'(x) = \frac{x}{|x|} e^{-|x|}$. Indeed let $\phi \in C_c^\infty(\mathbb{R})$.

Then supp $u \subset [-M, M]$ for some $M > 0$.

So $\int_{\mathbb{R}} u \phi' dx = \int_{-M}^{M} u \phi' dx$.

$$= \int_{-M}^{0} u \phi' dx + \int_{0}^{M} u \phi' dx.$$

$$= \int_{-M}^{0} e^{x} \phi' dx + \int_{0}^{M} e^{-x} \phi' dx.$$

$\underset{\text{by parts}}{\overset{\text{Integration}}{=}}$ $[e^{x}\phi]_{-M}^{0} - \int_{-M}^{0} (e^{x})' \phi dx + [e^{-x}\phi]_{0}^{M} + \int_{0}^{M} (e^{-x})' \phi dx$

$$= e^{0}\phi(0) - e^{-M} \underbrace{\phi(-M)}_{=0} + \underbrace{e^{-M} \phi(M)}_{=0} - e^{-0}\phi(0)$$

$$\int_{-M}^{0} (-e^{x}) \phi(x) dx + \int_{0}^{M} e^{-x} \phi(x) = \int_{-M}^{M} \frac{x}{|x|} e^{-|x|} \phi(x) dx$$

$= \int_{\mathbb{R}} \frac{x}{|x|} e^{-|x|} \phi(x)$ as desired.

Def 1.3 (Sobolev spaces) Let $k \in \mathbb{N}$.

Then

$$H^k(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n) : \partial^\alpha u \in L^2(\mathbb{R}^n), \forall \alpha \text{ with } |\alpha| \le k \}.$$

We equip $H^k(\mathbb{R}^n)$ with the norm $\| \, \|_{H^k(\mathbb{R}^n)}$ defined by

$$\|u\|_{H^k(\mathbb{R}^n)} := \left(\sum_{|\alpha| \le k} \|\partial^\alpha u\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

Ex 1.4 Let $u : \mathbb{R} \to \mathbb{R}$, $u(x) = e^{-|x|}$ then

$u \in L^2(\mathbb{R})$ und $u'(x) \underset{\text{Ex 1.3}}{=} \frac{x}{|x|} e^{-|x|} \Rightarrow u' \in L^2(\mathbb{R})$.

Thus, $u \in H^1(\mathbb{R})$.

(Exercise) Let u_n be as in example 0.2.

Then $u_n \longrightarrow u$ in $H^1(\mathbb{R})$, thus we have solved the issue arised in example 0.2. The following theorem shows that this is always the case.

Theorem 1.5 (Completeness) For any $k \in \mathbb{N}$ $(H^k(\mathbb{R}^n), \|\cdot\|_{H^k(\mathbb{R}^n)})$ is a Banach space.

Thm 1.6 (Approximation by smooth functions). Let $k \in \mathbb{N}$. If $u \in H^k(\mathbb{R}^n)$ then there exists a sequence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that $u_m \to u$ in $H^k(\mathbb{R}^n)$.

Thm 1.5 $\longrightarrow$ we have solved the completeness issue.

Thm 1.6 $\to H^k(\mathbb{R}^n)$ is the smallest complete extension of $H^k(\mathbb{R}^n) \cap C_c^\infty(\mathbb{R}^n)$ to it maintains a lot of their properties.