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For Exercise Sheet 4

Lemma: Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space, $T \in L(H)$, $G \subseteq H$ be a closed subspace of H, and P be the orthogonal projection on G. Then:
(i) $T(G) \subseteq G \;\Leftrightarrow\; PTP = TP$,
(ii) $T(G) \subseteq G$ and $T\left(G^{\perp}\right) \subseteq G^{\perp} \;\Leftrightarrow\; TP = PT \;\Leftrightarrow\; T(G) \subseteq G$ and $T^*(G) \subseteq G$.

Proof: (i) We have

$$\begin{aligned} T(G) \subseteq G \;&\Leftrightarrow\; \text{For all } y \in T(G) \text{ we have } y \in G = \text{range}(P) = \ker\left(I_H - P\right) \\ &\Leftrightarrow\; \text{For all } y \in T(G) \text{ we have } Py = y \\ &\Leftrightarrow\; \text{For all } y \in T(P(H)) \text{ we have } Py = y \\ &\Leftrightarrow\; \text{For all } x \in H \text{ we have } PTPx = TPx. \end{aligned}$$

(ii) Since P is the orthogonal projection on G, we have that

$$\text{range}\left(I_H - P\right) = \ker(P) \perp \text{range}(P) = \ker\left(I_H - P\right) = G,$$

i.e. $I_H - P$ is the orthogonal projection on $G^{\perp}$. By part (i) we obtain

$$\begin{aligned} T\left(G^{\perp}\right) \subseteq G^{\perp} \;&\Leftrightarrow\; \left(I_H - P\right) T \left(I_H - P\right) = T\left(I_H - P\right) \\ &\Leftrightarrow\; T - PT - TP + PTP = T - TP \\ &\Leftrightarrow\; PTP = PT. \end{aligned}$$

Then we get

$$T(G) \subseteq G \text{ and } T\left(G^{\perp}\right) \subseteq G^{\perp} \;\Leftrightarrow\; TP = PTP = PT.$$

For the other equivalence we use that P is self-adoint, and therefore by part (i)

$$\begin{aligned} T^*(G) \subseteq G \;&\Leftrightarrow\; PT^*P = T^*P \;\Leftrightarrow\; \left(PT^*P\right)^* = \left(T^*P\right)^* \\ &\Leftrightarrow\; PTP = PT \\ &\Leftrightarrow\; T\left(G^{\perp}\right) \subseteq G^{\perp}. \end{aligned}$$

□

Theorem: (Exponential decay of the Bessel potential kernel) Let $s > 0$. The kernel

$$G_s(x) := \mathcal{F}^{-1}\left(\left(1 + |\cdot|^2\right)^{-\frac{s}{2}}\right)(x),\; x \in \mathbb{R}^n,$$

is a smooth positive function and there are positive constants $C_{s,n}$ and $c_{s,n}$ such that

$$G_s(x) \le C_{s,n} \mathrm{e}^{-\frac{|x|}{2}} \text{ for } |x| \ge 2,$$

and such that

$$\frac{1}{c_{s,n}} \le \frac{G_s(x)}{H_s(x)} \le c_{s,n} \text{ for } |x| \le 2,$$

where H_s is

$$H_s(x) := \begin{cases} |x|^{s-n} + 1 + \mathcal{O}\left(|x|^{s-n+2}\right) & \text{for } s \in (0,n), \\ \log\left(\frac{2}{|x|}\right) + 1 + \mathcal{O}\left(|x|^2\right) & \text{for } s = n, \\ 1 + \mathcal{O}\left(|x|^{s-n}\right) & \text{for } s > n, \end{cases}$$

with $\mathcal{O}(t)$ such that $|\mathcal{O}(t)| \le C|t|$ for some positive constant $C > 0$ and for $t \to 0$.

End of the proof of the Theorem: We proved that for $x \in \mathbb{R}^n$ we have

$$G_s(x) = \frac{\sqrt{2}^{-n}}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty \mathrm{e}^{-t} \mathrm{e}^{-\frac{|x|^2}{4t}} t^{\frac{s-n}{2}-1} \mathrm{d}t > 0,$$

G_s is smooth on $\mathbb{R}^n \backslash \{0\}$ and for $x \in \mathbb{R}^n$ with $|x| \geq 2$ there is some constant $C_{s,n} > 0$ such that

$$G_s(x) \leq C_{s,n} \mathrm{e}^{-\frac{|x|}{2}}.$$

Now for $x \in \mathbb{R}^n$ with $|x| \leq 2$ we decompose G_s in three parts:

$$\begin{aligned}
G_s^{(1)}(x) &= \frac{\sqrt{2}^{-n}}{\Gamma\left(\frac{s}{2}\right)} \int_0^{|x|^2} \mathrm{e}^{-t} \mathrm{e}^{-\frac{|x|^2}{4t}} t^{\frac{s-n}{2}-1} \mathrm{d}t \\
&= |x|^{s-n} \frac{\sqrt{2}^{-n}}{\Gamma\left(\frac{s}{2}\right)} \int_0^1 \mathrm{e}^{-|x|^2 t} \mathrm{e}^{-\frac{1}{4t}} t^{\frac{s-n}{2}-1} \mathrm{d}t, \\
G_s^{(2)}(x) &= \frac{\sqrt{2}^{-n}}{\Gamma\left(\frac{s}{2}\right)} \int_{|x|^2}^4 \mathrm{e}^{-t} \mathrm{e}^{-\frac{|x|^2}{4t}} t^{\frac{s-n}{2}-1} \mathrm{d}t, \\
G_s^{(3)}(x) &= \frac{\sqrt{2}^{-n}}{\Gamma\left(\frac{s}{2}\right)} \int_4^{\infty} \mathrm{e}^{-t} \mathrm{e}^{-\frac{|x|^2}{4t}} t^{\frac{s-n}{2}-1} \mathrm{d}t, \\
\text{i.e. } G_s(x) &= G_s^{(1)}(x) + G_s^{(2)}(x) + G_s^{(3)}(x).
\end{aligned}$$

We now look at the three parts separatly:
$G_s^{(1)}$: It is for $t \in (0,1)$ and $x \in \overline{B_2(0)}$:

$$\mathrm{e}^{-t|x|^2} = \sum_{k=0}^{\infty} \frac{\left(-t|x|^2\right)^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{\left(-t|x|^2\right)^k}{k!} = 1 + \mathcal{O}\left(t|x|^2\right),$$

since $t|x|^2 \leq 4$. Therefore we get that

$$\begin{aligned}
G_s^{(1)}(x) &= |x|^{s-n} \frac{\sqrt{2}^{-n}}{\Gamma\left(\frac{s}{2}\right)} \int_0^1 \mathrm{e}^{-\frac{1}{4t}} t^{\frac{s-n}{2}-1} \mathrm{d}t + \mathcal{O}\left(|x|^{s-n+2}\right) \int_0^1 \mathrm{e}^{-\frac{1}{4t}} t^{\frac{s-n}{2}} \mathrm{d}t \\
&= c_{s,n}^{(1)} |x|^{s-n} + \mathcal{O}\left(|x|^{s-n+2}\right)
\end{aligned}$$

for $x \in \overline{B_2(0)}$.
$G_s^{(2)}$: It follows for $x \in \overline{B_2(0)}$, $t \in \left(|x|^2, 4\right)$ from $0 \leq \frac{|x|^2}{4t} \leq \frac{1}{4}$ that

$$\mathrm{e}^{-\frac{17}{16}} = \mathrm{e}^{-4-\frac{1}{4}} \leq \mathrm{e}^{-t-\frac{|x|^2}{4t}} \leq 1.$$

Therefore

$$\mathrm{e}^{-\frac{17}{16}} \int_{|x|^2}^4 t^{\frac{s-n}{2}-1} \mathrm{d}t \leq G_s^{(2)}(x) \leq \int_{|x|^2}^4 t^{\frac{s-n}{2}-1} \mathrm{d}t$$

and

$$\int_{|x|^2}^4 t^{\frac{s-n}{2}-1} \mathrm{d}t = \begin{cases} \frac{2}{n-s}|x|^{s-n} - \frac{2^{s-n+1}}{n-s}, & \text{for } s < n, \\ 2\log\left(\frac{2}{|x|}\right), & \text{for } s = n, \\ \frac{2^{s-n+1}}{s-n} - \frac{2}{s-n}|x|^{s-n}, & \text{for } s > n. \end{cases}$$

$G_s^{(3)}$: For $x \in \overline{B_2(0)}$ and $t \in (4, \infty)$ we have

$$0 < \mathrm{e}^{-\frac{1}{4}} = \mathrm{e}^{-\frac{1}{4} \cdot \frac{2^2}{4}} \leq \mathrm{e}^{-\frac{|x|^2}{4t}} \leq 1$$

and thus

$$0 < c_{s,n}^{(3)} \leq G_s^{(3)} \leq \tilde{c}_{s,n}^{(3)}.$$

Note that for $s > n$ we can estimate

$$|x|^{s-n+2} \leq 2^{s-n+2} \text{ for } x \in \overline{B_2(0)}$$

and so $|x|^{s-n+2} \in \mathcal{O}(1)$. □

Example ("Fixpoint" of the Fourier Transform): It is

$$\mathcal{F}\left(\mathrm{e}^{-|\cdot|^2}\right)(\xi) = \sqrt{2}^{-n} \mathrm{e}^{-\frac{|\xi|^2}{4}} \text{ for all } \xi \in \mathbb{R}^n.$$

Proof: Define $f(x) := \mathrm{e}^{-|x|^2}$ for $x \in \mathbb{R}^n$. First we prove this for $n = 1$: It is

$$\widehat{f}(\xi) := \mathcal{F}(f)(\xi) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi x} \mathrm{e}^{-x^2} \mathrm{d}x \text{ for all } \xi \in \mathbb{R}.$$

Then we get for the first derivative of $\widehat{f}$ by integration by parts:

$$\begin{aligned}
\widehat{f}'(\xi) &= -\mathrm{i}(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} x\mathrm{e}^{-\mathrm{i}\xi x} \mathrm{e}^{-x^2} \mathrm{d}x \\
&= \left[\mathrm{i}\frac{1}{2}(2\pi)^{-\frac{1}{2}} \mathrm{e}^{-x^2} \mathrm{e}^{-\mathrm{i}\xi x}\right]_{-\infty}^{\infty} - \mathrm{i}\frac{1}{2}(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-x^2}(-\mathrm{i}\xi)\mathrm{e}^{-\mathrm{i}\xi x} \mathrm{d}x \\
&= -\frac{\xi}{2}(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\xi x} \mathrm{e}^{-x^2} \mathrm{d}x = -\frac{\xi}{2}\widehat{f}(\xi)
\end{aligned}$$

for all $\xi \in \mathbb{R}$. By ODE-Theory we know that

$$\widehat{f}(\xi) = c\mathrm{e}^{-\frac{\xi^2}{4}} \text{ for all } \xi \in \mathbb{R}$$

for some constant $c \in \mathbb{R}$. It is

$$c = \widehat{f}(0) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-x^2} \mathrm{d}x = \frac{1}{\sqrt{2}\sqrt{\pi}}\sqrt{\pi} = \frac{1}{\sqrt{2}}.$$

It follows:

$$\widehat{f}(\xi) = \sqrt{2}^{-1} \mathrm{e}^{-\frac{\xi^2}{4}},\ \xi \in \mathbb{R}.$$

For arbitrary $n \in \mathbb{N}$:

$$\begin{aligned}
\widehat{f}(\xi) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\xi\cdot x} \mathrm{e}^{-|x|^2} \mathrm{d}x \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\sum\limits_{j=1}^{n} \xi_j x_j} \mathrm{e}^{-\sum\limits_{j=1}^{n} x_j^2} \mathrm{d}x \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \prod_{j=1}^{n} \mathrm{e}^{-\mathrm{i}\xi_j x_j} \mathrm{e}^{-x_j^2} \mathrm{d}x \\
&= \prod_{j=1}^{n} \left[(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\xi_j x_j} \mathrm{e}^{-x_j^2} \mathrm{d}x_j\right] \\
&= \prod_{j=1}^{n} \left[\sqrt{2}^{-1} \mathrm{e}^{-\frac{\xi_j^2}{4}}\right] \\
&= \sqrt{2}^{-n} \mathrm{e}^{-\sum\limits_{j=1}^{n} \frac{\xi_j^2}{4}} \\
&= \sqrt{2}^{-n} \mathrm{e}^{-\frac{|\xi|^2}{4}}
\end{aligned}$$

for all $\xi \in \mathbb{R}^n$. $\square$