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Spectral theory

1. Exercise Sheet - Solutions

Exercise 1 (Closed Operators)

1. Show that every separable Hilbertspace H has a finite or countable orthonormal basis.
2. We consider on the Hilbertspace $H = L^2(0,1)$ the linear operator T acting on the Sobolevspace $D(T) = H^1(0,1)$ by $Tf = \mathrm{i}f'$. Is the operator T closed? / symmetric? / self-adjoint? / semi-bounded from below?
3. Now we consider again the Hilbertspace $H = L^2(0,1)$ and the linear operator $\widetilde{T}$ acting on the Sobolevspace $D\left(\widetilde{T}\right) = H_0^1(0,1)$ by $\widetilde{T}f = \mathrm{i}f'$. Is the operator $\widetilde{T}$ closed? / symmetric? / self-adjoint? / semi-bounded from below?

Exercise 2 (Multiplication operators are closed)

Consider the Hilbertspace $H = L^2\left(\mathbb{R}^d\right)$ and fix some function $f \in L^\infty_{\text{loc}}\left(\mathbb{R}^d\right)$. We introduce the multiplication operator M_f acting on

$$D\left(M_f\right) = \left\{u \in L^2\left(\mathbb{R}^d\right) : fu \in L^2\left(\mathbb{R}^d\right)\right\} \text{ by } M_f u = fu.$$

Show that the linear operator M_f is closed in H.

Solution of Exercise 2

Let $(x_n)_{n\in\mathbb{N}} \subseteq D\left(M_f\right)$ be a Cauchy-sequence w.r.t. $\|\cdot\|_{M_f}$-Norm. Since the Lebesgue space $L^2\left(\mathbb{R}^d\right)$ is complete there are two functions $u, v \in L^2\left(\mathbb{R}^d\right)$ with

$$\lim_{n\to\infty} u_n = u, \ \lim_{n\to\infty} fu_n = v \text{ in } L^2\left(\mathbb{R}^d\right).$$

Then we have for any $n \in \mathbb{N}$:

$$\begin{aligned}
\|fu - v\|^2_{L^2(B_n(0))} &= \int_{B_n(0)} |f(x)u(x) - v(x)|^2 \,\mathrm{d}x = \lim_{n\to\infty} \int_{B_n(0)} |f(x)u(x) - f(x)u_n(x)|^2 \,\mathrm{d}x \\
&\le \|f\|^2_{L^\infty\left(\overline{B_n(0)}\right)} \|u - u_n\|^2_{L^2(\mathbb{R}^d)} \text{ by Hölder-inequality} \\
&\to 0 \text{ as } n \to \infty,
\end{aligned}$$

i.e. $fu = v$ almost everywhere on $B_n(0) \subseteq \mathbb{R}^d$ for all $n \in \mathbb{N}$. This implies now directly that

$$fu = v \text{ almost everywhere on } \mathbb{R}^d.$$

So $fu = v \in L^2\left(\mathbb{R}^d\right)$, i.e. $u \in D\left(M_f\right)$ with

$$M_f u = v = fu.$$

This shows that the operator M_f is closed. □