12.05.2017
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Spectral theory

2. Exercise Sheet - Solutions

Exercise 1 (Proof of Lemma IV.2)

1. (Lemma of Fekete) If $(a_n)_{n\in\mathbb{N}} \subseteq [0,\infty)$ is a real sequence with $0 \le a_{n+m} \le a_n \cdot a_m$ for all $n,m \in \mathbb{N}$, then there holds
$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} a_n^{\frac{1}{n}}.$$

2. We define for a bounded operator $A \in L(X)$ the spectral radius
$$r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}.$$
Show the following:
$$r(A) = \lim_{n\to\infty} \|A^n\|_{L(X)}^{\frac{1}{n}}.$$

3. Let $A \in L(X)$ be a bounded operator and $\lambda \in \mathbb{C}$ such that $|\lambda| > r(A)$. Show that
$$R(\lambda, A) = \sum_{n=0}^{\infty} \lambda^{-(n+1)} A^n.$$

4. Let $A \in L(X)$ be a bounded operator. Assume that $\lambda \in \mathbb{C}$ with $|\lambda| > \delta \|A\|_{L(X)}$ for some $\delta > 1$, show in this case the estimate
$$\|\lambda R(\lambda, A)\|_{L(X)} < \frac{\delta}{\delta - 1}.$$

Solution of Exercise 1

(1): Let $(a_n)_{n\in\mathbb{N}} \subseteq [0,\infty)$ be such that
$$0 \le a_{n+m} \le a_n \cdot a_m \text{ for all } n,m \in \mathbb{N}.$$
Since it is $a_n^{\frac{1}{n}} \ge 0$ for all $n \in \mathbb{N}$, the infimum $a := \inf_{n\in\mathbb{N}} a_n^{\frac{1}{n}} \in [0,\infty)$ exists. Let $\varepsilon > 0$ be arbitrary, so choose some $n_0 \in \mathbb{N}_{\ge 2}$ with
$$0 \le a_{n_0}^{\frac{1}{n_0}} < a + \varepsilon.$$
For any $m \in \mathbb{N}$ we can choose a unique $r_m \in \{1, \dots, n_0 - 1\}$, $t_m \in \mathbb{N}$ with
$$m = t_m n_0 + r_m.$$
Therefore we get
$$\begin{aligned} 0 \le a_m &= a_{t_m n_0 + r_m} \le a_{t_m n_0} \cdot a_{r_m} \le a_{n_0}^{t_m} \cdot a_{r_m} < (a+\varepsilon)^{t_m n_0} a_{r_m} \\ &= (a+\varepsilon)^{t_m n_0 + r_m} (a+\varepsilon)^{-r_m} a_{r_m} = (a+\varepsilon)^m \max_{n\in\{1,\dots,n_0\}} \Big((a+\varepsilon)^{-n} a_n\Big). \end{aligned}$$
For the limit we get
$$a \le a_m^{\frac{1}{m}} \le (a+\varepsilon) \max_{n\in\{1,\dots,n_0\}} \Big((a+\varepsilon)^{-n} a_n\Big)^{\frac{1}{m}} \overset{m\to\infty}{\longrightarrow} (a+\varepsilon) \overset{\varepsilon\to 0^+}{\longrightarrow} a,$$

i.e.

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = a.$$

(3 for $|\lambda| > \|A\|_{L(X)}$): Let $\lambda \in \mathbb{C}$ such that $|\lambda| > \|A\|_{L(X)}$. It is

$$(\lambda - A) = \lambda \left(\mathrm{Id}_X - \lambda^{-1} A\right).$$

We define $S := \lambda^{-1} A \in L(X)$, i.e. $\|S\|_{L(X)} < 1$. By Neumann-series:

$$\left(\mathrm{Id}_X - \lambda^{-1} A\right)^{-1} = \sum_{n=0}^{\infty} \left(\lambda^{-1} A\right)^n = \sum_{n=0}^{\infty} \lambda^{-n} A^n$$

$$\Rightarrow \; (\lambda - A)^{-1} = \lambda^{-1} \left(\mathrm{Id}_X - \lambda^{-1} A\right)^{-1} = \sum_{n=0}^{\infty} \lambda^{-(n+1)} A^n.$$

(2) We define for $n \in \mathbb{N}$

$$a_n := \|A^n\|_{L(X)} \in [0, \infty).$$

Then by sub-multiplicativity we get that

$$0 \leq a_{n+m} = \left\|A^{n+m}\right\|_{L(X)} = \|A^n A^m\|_{L(X)} \leq \|A^n\|_{L(X)} \|A^m\|_{L(X)} = a_n \cdot a_m \text{ for all } n, m \in \mathbb{N}.$$

Therefore the Lemma of Fekete implies the existence of

$$a := \lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} \|A^n\|_{L(X)}^{\frac{1}{n}} \in [0, \infty).$$

"$\leq$": Let $\lambda \in \mathbb{C}$ with $|\lambda| > a$, then we have

$$\sum_{n=0}^{\infty} \left\|\lambda^{-(n+1)} A^n\right\|_{L(X)} = \sum_{n=0}^{\infty} |\lambda|^{-(n+1)} \|A^n\|_{L(X)} < \infty,$$

since

$$\limsup_{n\to\infty} \sqrt[n]{|\lambda|^{-(n+1)} \|A^n\|_{L(X)}} = \limsup_{n\to\infty} |\lambda|^{-1} |\lambda|^{-\frac{1}{n}} \|A^n\|_{L(X)}^{\frac{1}{n}} = |\lambda|^{-1} a < 1.$$

We define $R_\lambda := \sum\limits_{n=0}^{\infty} \lambda^{-(n+1)} A^n$ and R_λ converge absolutly with

$$(\lambda - A) R_\lambda = R_\lambda (\lambda - A) = \mathrm{Id}_X.$$

Hence $(\lambda - A)^{-1} \in L(X)$ exists and $\lambda \in \rho(A)$, i.e. $r(A) \leq a$.
"$\geq$": Let $\lambda \in \mathbb{C}$ with $|\lambda\rangle r(A)$. For $x' \in X'$ we define the map

$$L(\mu) := x' \left(R(\mu, A)\right), \;\; \mu \in \rho(A).$$

Then we know that the map $L\colon \rho(A) \to \mathbb{C}$ is analytic for $\mu \in \mathbb{C}$ with $|\mu| > \|A\|_{L(X)}$ with power serie

$$L(\mu) = \sum_{n=0}^{\infty} x' \left(A^n\right) \mu^{-(n+1)}.$$

Since analytic functions have the same serie representation on the whole domain we get

$$L(\mu) = \sum_{n=0}^{\infty} x' \left(A^n\right) \mu^{-(n+1)} \text{ for } \mu \in \rho(A).$$

Then the set $\left\{\lambda^{-(n+1)} x' \left(A^n\right) : n \in \mathbb{N}\right\} \subseteq \mathbb{C}$ is bounded for all $x' \in X'$. By Banach-Steinhaus Theorem also the set

$$\left\{\lambda^{-(n+1)} A^n \colon n \in \mathbb{N}\right\} \subseteq L(X)$$

is bounded by some constant $C > 0$. This implies directly that

$$\|A^n\|_{L(X)} \leq C \, |\lambda|^{n+1}.$$

And so we have

$$a = \lim_{n\to\infty} \|A^n\|_{L(X)}^{\frac{1}{n}} \leq \lambda \text{ and } r(A) \geq a.$$

(3) The proof of (2) shows (3).
(4) Let $\lambda \in \mathbb{C}$ with $|\lambda| > \delta \|A\|_{L(X)}$, i.e. $\lambda \in \rho(A)$. So we have

$$\lambda R(\lambda, A) = \sum_{n=0}^{\infty} \lambda^{-n} A^n.$$

This implies by the geometric sum:

$$\begin{aligned}\|\lambda R(\lambda, A)\|_{L(X)} &\leq \sum_{n=0}^{\infty} |\lambda|^{-n} \|A\|_{L(X)}^n < \sum_{n=0}^{\infty} \left(\delta \|A\|_{L(X)}\right)^{-n} \|A\|_{L(X)}^n \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{\delta}\right)^n = \frac{1}{1-\frac{1}{\delta}} = \frac{\delta}{\delta - 1}.\end{aligned}$$

□

Exercise 2 (Dunford Lemma)

Show that weakly analytic is equivalent to analytic.