Spectral theory

3. Exercise Sheet

**Definition** (Fréchet-Differentiable): Let \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) be two Banach spaces, and \(f : X \to Y\) a map. The function \(f\) is Fréchet-differentiable iff there exists an operator \(A \in L(X,Y)\) with
\[
\frac{1}{\|h\|_X} \|f(x+h) - f(x) - Ah\|_Y \to 0 \text{ as } \|h\|_X \to 0.
\]

**Exercise 1**

Let \(A \in L(X)\), \(\Omega' \subseteq \mathbb{C}\) be open and bounded with \(\sigma(A) \subseteq \Omega'\) and \(\emptyset \neq I \subseteq \mathbb{R}\) open, \(f : I \to H^\infty(\Omega')\), \(t \mapsto f(t, \cdot)\). Then:

i) If \(t \mapsto f(t, \cdot)\) is Fréchet-differentiable, then the map \(t \mapsto f(t, A)\) is differentiable with
\[
\frac{d}{dt}f(t,A) = \left(\frac{d}{dt}f(t, \cdot)\right)(A).
\]

ii) If \([a, b] \ni t \mapsto f(t, \cdot) \in H^\infty(\Omega')\) is continuous and \(g(\cdot) := \int_{a}^{b} f(t, \cdot)dt\), then the map \(t \mapsto f(t, A)\) is continuous with
\[
g(A) = \int_{a}^{b} f(t, A)dt.
\]

We define the set
\[
H^\infty(\Omega') := \{f : \Omega' \to \mathbb{C} \mid f \text{ is analytic and bounded on } \Omega\}.
\]

**Exercise 2** (Logarithm of a bounded operator)

Let \(A \in L(X)\) with \((-\infty, 0] \subseteq \rho(A)\), and define the function
\[
f : \mathbb{R} \times \mathbb{C}\setminus(-\infty, 0] \to \mathbb{C}, \ (t, z) \mapsto -iz^t.
\]
Calculate:
\[
\frac{d}{dt}f(t, A)|_{t=0}.
\]
(Hint: Use Exercise 1.)

**Exercise 3** (Commuting bounded operators)

Let \(A, B \in L(X)\) be such that \(AB = BA\) and \(f \in H(\sigma(A)) := \{f : U \to \mathbb{C} \mid U \subseteq \mathbb{C} \text{ open, } \sigma(A) \subseteq U, \ f \text{ is analytic on } U\}\). Then it holds
\[
Bf(A) = f(A)B.
\]

**Exercise 4** (Composition of Dunford maps)

If we have \(A \in L(X), f \in H(\sigma(A)), g \in H(f(\sigma(A)))\), then it is \(g \circ f \in H(\sigma(A))\) with
\[
(g \circ f)(A) = g(f(A)).
\]