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Spectral theory

3. Exercise Sheet

Definition (Fréchet-Differentiable): Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two Banach spaces, and $f\colon X \to Y$ a map. The function f is Fréchet-differentiable iff there exists an operator $A \in L(X,Y)$ with

$$\frac{1}{\|h\|_X} \|f(x+h) - f(x) - Ah\|_Y \to 0 \text{ as } \|h\|_X \to 0.$$

Exercise 1 ()

Let $A \in L(X)$, $\Omega' \subseteq \mathbb{C}$ be open and bounded with $\sigma(A) \subseteq \Omega'$ and $\emptyset \neq I \subseteq \mathbb{R}$ open, $f\colon I \to H^\infty(\Omega')$, $t \mapsto f(t,\cdot)$. Then:

i) If $t \mapsto f(t,\cdot)$ is Fréchet-differentiable, then the map $t \mapsto f(t,A)$ is differentiable with

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t,A) = \left(\frac{\mathrm{d}}{\mathrm{d}t} f(t,\cdot)\right)(A).$$

ii) If $[a,b] \ni t \mapsto f(t,\cdot) \in H^\infty(\Omega')$ is continuous and $g(\cdot) := \int_a^b f(t,\cdot)\mathrm{d}t$, then the map $t \mapsto f(t,A)$ is continuous with

$$g(A) = \int_a^b f(t,A)\mathrm{d}t.$$

We define the set

$$H^\infty(\Omega') := \{f\colon \Omega' \to \mathbb{C} \mid f \text{ is analytic and bounded on } \Omega\}.$$

Exercise 2 (Logarithm of a bounded operator)

Let $A \in L(X)$ with $(-\infty, 0] \subseteq \rho(A)$, and define the function

$$f\colon \mathbb{R} \times \mathbb{C}\backslash(-\infty,0] \to \mathbb{C},\ (t,z) \mapsto -\mathrm{i}z^{\mathrm{i}t}.$$

Calculate:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t,A)|_{t=0}.$$

(Hint: Use Exercise 1.)

Exercise 3 (Commuting bounded operators)

Let $A, B \in L(X)$ be such that $AB = BA$ and $f \in H(\sigma(A)) := \{f\colon U \to \mathbb{C} \mid U \subseteq \mathbb{C} \text{ open}, \sigma(A) \subseteq U,\ f \text{ is analytic on } U\}$. Then it holds

$$Bf(A) = f(A)B.$$

Exercise 4 (Composition of Dunford maps)

If we have $A \in L(X)$, $f \in H(\sigma(A))$, $g \in H(f(\sigma(A)))$, then it is $g \circ f \in H(\sigma(A))$ with

$$(g \circ f)(A) = g(f(A)).$$