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Spectral theory

4. Exercise Sheet

Exercise 1 (Sobolev embedding theorem)

Let $s = \frac{n}{2}$. Show that: The Sobolev space $H^s(\mathbb{R}^n)$ continuously embeds in the $L^q(\mathbb{R}^n)$ space for any $2 < q < \infty$. (Hint: Use the Theorem below)

In the Exercise class I will show the following Theorem:
Theorem: (Exponential decay of the Bessel potential kernel) Let $s > 0$. The kernel

$$G_s(x) := \mathcal{F}^{-1}\left(\left(1 + |\cdot|^2\right)^{-\frac{s}{2}}\right)(x), \ x \in \mathbb{R}^n,$$

is a smooth positive function and there are positive constants $C_{s,n}$ and $c_{s,n}$ such that

$$G_s(x) \leq C_{s,n}\mathrm{e}^{-\frac{|x|}{2}} \text{ for } |x| \geq 2,$$

and such that

$$\frac{1}{c_{s,n}} \leq \frac{G_s(x)}{H_s(x)} \leq c_{s,n} \text{ for } |x| \leq 2,$$

where H_s is

$$H_s(x) := \begin{cases} |x|^{s-n} + 1 + \mathcal{O}\left(|x|^{s-n+2}\right) \text{ for } s \in (0,n), \\ \log\left(\frac{2}{|x|}\right) + 1 + \mathcal{O}\left(|x|^2\right) \text{ for } s = n, \\ 1 + \mathcal{O}\left(|x|^{s-n}\right) \text{ for } s > n, \end{cases}$$

with $\mathcal{O}(t)$ such that $|\mathcal{O}(t)| \leq C|t|$ for some positive constant $C > 0$ and for $t \to 0$.

Exercise 2 ()

Let $A\colon D(A) \subseteq H \to H$ be a self-adjoint operator on a Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$. Show that:

$$\sigma_{\text{ess}}(A) = \emptyset \ \Leftrightarrow \ \text{the operator } A \text{ has a compact resolvent.}$$

Exercise 3 (Missing part in the Kato-Rellich Theorem)

Let $A\colon D(A) \subseteq H \to H$ be a self-adjoint operator on a Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ and $B\colon D(B) \subseteq H \to H$ be a symmetric A-bounded operator with a relative bound less than 1. Show that, if the operator A is essentially self-adjoint on some domain $D \subseteq D(A)$, then also the operator $A + B$ is essentially self-adjoint on D.