Institut für Analysis
PD Dr. Peer Christian Kunstmann
Dr. Michal Jex

SS2018/19
29.4.2019

Spectral Theory

1st Exercise Sheet

Exercise 1: (Prove the following claim)
Let X, Y be Banach spaces and $A : \mathcal{D}(A) \subseteq X \to Y$ be a closed operator. Furthermore assume that $r : [a, b] \to X$ is continuous with $r(t) \in \mathcal{D}(A)$ and $Ar : [a, b] \to Y$ is continuous for all $t \in [a, b]$ then

$$\int_a^b r(t)\mathrm{d}t \in \mathcal{D}(A) \text{ and } A\int_a^b r(t)\mathrm{d}t = \int_a^b Ar(t)\mathrm{d}t\,.$$

Exercise 2:
Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a domain, $\alpha \in \mathbb{N}_0^d$ be a multi-index and $1 \leq p, q \leq \infty$. Show that the operator of a weak derivative

$$\mathscr{D} = \frac{\partial^\alpha}{\partial x^\alpha} : \mathcal{D}(\mathscr{D}) := \{u \in L^p(\Omega) : \mathscr{D}u \in L^q(\Omega)\} \subseteq L^p(\Omega) \to L^q(\Omega)$$

is linear, densely defined and closed.

Exercise 3:
Let X be a Banach space, $f : [a, b] \to X$ and $\Phi : [a, b] \to X'$. Show the following:

1. If $t \mapsto \langle f(t), \psi\rangle \in C^1[a, b]$ for all $\psi \in X'$ then $f \in C([a, b], X)$.
2. If $t \mapsto \langle g, \Phi(t)\rangle \in C^1[a, b]$ for all $g \in X$ then $\Phi \in C([a, b], X)$.
3. If $t \mapsto \langle f(t), \psi\rangle \in C^{k+1}[a, b]$ for all $\psi \in X'$ then $f \in C^k([a, b], X)$ for every $k \in \mathbb{N}$.
4. If $t \mapsto \langle g, \Phi(t)\rangle \in C^{k+1}[a, b]$ for all $g \in X$ then $\Phi \in C^k([a, b], X)$ for every $k \in \mathbb{N}$.

Exercise 4:
Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a domain and $X := (C_b^0(\Omega), \|\cdot\|_\infty)$ be a set of all bounded continuous functions equipped with a supremum norm. For fixed $m \in X$ define

$$M_m : X \to X, f \mapsto m \cdot f\,.$$

Calculate the following:

1. spectrum of M_m and
2. resolvent function of M for every $\lambda \in \rho(M_m)$.

Exercise 5:
Let $X = (C[0, 1], \|\cdot\|_\infty)$. For fixed $k \in C([0, 1] \times [0, 1])$ define the operator $T : X \to X$ as

$$(Tf)(t) := \int_0^t k(t, s)f(s)\mathrm{d}s\,.$$

for every $f \in X$ and $t \in [0, 1]$. Show the following:

1. T is well defined,
2. $\|T\| \leq \sup_{t\in[0,1]} \int_0^t |k(t, s)|\mathrm{d}s$ and
3. $\sigma(T) = \{0\}$.