Karlsruher Institut für Technologie
Institut für Analysis
PD Dr. Peer Christian Kunstmann
Dr. Michal Jex

SS2018/19
6.5.2019

Spectral Theory

2nd Exercise Sheet

Exercise 6:
Let A be a closed linear operator in X.

1. Let X be reflexive, $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(A)$ such that $x_n \to x$ in X and $\sup_n \|Ax_n\| < \infty$. Show that $x \in \mathcal{D}(A)$.
2. This is false, in general, if X is not reflexive. Show this for the following:
   Let $X = C([0,1])$ and $A = \frac{\mathrm{d}}{\mathrm{d}x}$ with the domain $\mathcal{D}(A) = C^1([0,1])$.

Exercise 7:
Let A be a linear operator from X to Y. Show that the following are equivalent:

1. The operator A is closable,
2. $\overline{\mathrm{gr}(A)}$ is a graph of a linear operator from X to Y,
3. if x_n is a sequence in $\mathcal{D}(A)$ such that $x_n \to 0$ in X and $Ax_n \to y$ in Y then $y = 0$.

Exercise 8:
Let X be a Banach space and $\Phi : \mathcal{D}(\Phi) \subseteq X \to \mathbb{C}$ be linear and $\mathcal{D}(\Phi)$ be dense in X. Then

$$\Phi \text{ closable } \Leftrightarrow \Phi \text{ bounded}.$$

Exercise 9:
1. Let $m \in L^\infty(\mathbb{R})$ be fixed. We define the operator $(Sf)(t) := m(t)f(t+1)$ for $f \in L^1(\mathbb{R})$ and $t \in \mathbb{R}$ a.e. Show that the operator S is bounded and determine $S' \in \mathcal{L}(L^\infty(\mathbb{R}))$.
2. Let $T : L^1(0,1) \to c_0$ be defined as $(Tf)_n = \int_0^1 f(t)t^n \mathrm{d}t$. Show that the operator T is linear, bounded and determine T'.
3. Let $X = L^2(0,1)$ and $A = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with the domain $\mathcal{D}(A) = C_c^\infty(0,1)$. Show that $W^{2,2}(0,1) \subseteq \mathcal{D}(A')$, $A'g = g''$ for $g \in W^{2,2}(0,1)$ and that A is closable.