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Spectral Theory

5th Exercise Sheet

Exercise 16: Show the following:
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $T \in \mathscr{L}(X,Y)$ be a Fredholm operator. Then $T' \in \Phi(Y', X')$ and $\operatorname{ind}(T) = -\operatorname{ind}(T')$.

Exercise 17: Show the following:
Let A be a closed operator in a Banach space X, $\lambda_0 \in \rho(A)$ and $z \in \mathbb{C} \setminus \{\lambda_0\}$. Then for all $k \in \mathbb{N}$ we have

$$N\left((z-A)^k\right) = N\left(\left((\lambda_0 - z)^{-1} - R(\lambda_0, A)\right)^k\right),\ R\left((z-A)^k\right) = R\left(\left((\lambda_0 - z)^{-1} - R(\lambda_0, A)\right)^k\right).$$

In particular, $z - A \in \Phi([D(A)], X)$ if and only if $(\lambda_0 - z)^{-1} - R(\lambda_0, A) \in \Phi(X)$. In this case one has $\operatorname{ind}(z - A) = \operatorname{ind}((\lambda_0 - z)^{-1} - R(\lambda_0, A))$.

Exercise 18:

1. Let R, L be a right and left shift operator respectively defined on $l^2(\mathbb{N})$. For which $\lambda \in \mathbb{C}$ one has $\lambda - R \in \Phi(l^2)$ and $\lambda - L \in \Phi(l^2)$. What are $\operatorname{ind}(\lambda - L)$ and $\operatorname{ind}(\lambda - R)$ for such λ respectively?
2. Let $A = \frac{d^2}{dx^2}$ and consider $A : C^2[0,1] \to C[0,1]$. For which $\lambda \in \mathbb{C}$ the operator $\lambda - A$ is Fredholm? Determine $\operatorname{ind}(\lambda - A)$ for these λ.