**Spectral Theory**

6th Exercise Sheet

**Exercise 19:**
Let $X, Y, Z$ be Banach spaces, $T \in \Phi(X,Y)$ and $S \in \Phi(Y,Z)$. Show that $ST \in \Phi(X,Z)$ and $\text{ind}(ST) = \text{ind}(S) + \text{ind}(T)$.

*Hint:* First find a complement $N$ of $N(T)$ in $N(ST)$ and a complement $N_S$ of $T(N)$ in $N(S)$. Then represent $X$ and $Y$ as direct sums.

**Exercise 20:** Show the following:
Let $A$ be a linear operator in $\mathbb{C}^n$ given by $A = \begin{pmatrix} \lambda & 1 \\ & \ddots & \ddots \\ & & \lambda & 1 \end{pmatrix}$ where $\lambda \in \mathbb{C}$. Find a representation for the operator $f(A)$ (given by the Dunford functional calculus), where $f$ is holomorphic on a neighborhood of $\lambda$.

**Exercise 21:**
Let $X = L^2(\mathbb{R}^+)$ and let $T : X \to X$ be defined by

$$Tf(x) := \frac{1}{x} \int_0^x f(y) \, dy \quad \text{for} \quad f \in X, x \in \mathbb{R}^+$$

Show that:
1. $T$ is well defined, linear and continuous.
2. $T$ is not compact.

*Hint:* You can use Fredholm alternative.