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Spectral Theory

9th Exercise Sheet

Exercise 29: *Polar decomposition for bounded operators*
Show that to any $B \in \mathcal{L}(H)$ there is just one partial isometry W_B such that $B = W_B|B|$ and $N(W_B) = N(B)$. Furthermore, the identity $R(W_B) = \overline{R(B)}$ holds.

Exercise 30:
Show the following:
An operator $B \in \mathcal{L}(H)$ belongs to the trace class if and only if it is a product of two Hilbert–Schmidt operators.
Hint: Use the previous exercise.

Exercise 31:
Let $W \in S_1(H)$. Then W is called a statistical operator if it is positive and fulfills the normalization condition $\operatorname{Tr} W = 1$. Show the following:

1. The set $\mathcal{W}$ of all statistical operators on a given H is convex.
2. An operator $W \in \mathcal{W}$ is one–dimensional projection if and only if it is an extremal point of $\mathcal{W}$, i.e., if and only if the condition $W = \alpha W_1 + (1-\alpha)W_2$ with $0 < \alpha < 1$ and $W_1, W_2 \in \mathcal{W}$ implies $W_1 = W_2 = W$.

Exercise 32:
Let $S \in L(H)$ be self-adjoint. Furthermore let $\Psi : B_b(\sigma(S)) \to \mathcal{L}(H)$ be the functional calculus from 13.6. Then for $\lambda \in \sigma(S)$ one has

$$\psi(1_{\{\lambda\}}) = 0 \Leftrightarrow \lambda \notin \sigma_p(S)\,.$$

More precisely, $\psi(1_{\{\lambda\}})$ is the orthogonal projection onto $N(\lambda - S)$.