Spectral Theory
10th Exercise Sheet

Exercise 33:
Let $A$ be a self-adjoint positive operator. Then we define for $t \in \mathbb{R}^+$

$$T(t) := e^{-tA}.$$ 

Show that
1. $T(t)$ is self-adjoint.
2. $T(t)$ is a strongly continuous semigroup, i.e. $T(t)T(s) = T(t+s)$ for every $s, t \in \mathbb{R}^+$.
3. Furthermore let $x \in \mathcal{D}(A)$ and $Ax = y$. Then for every $s > 0$

$$\lim_{t \to 0^+} \frac{1}{t} (T(s)x - T(t+s)x) = T(s)y$$

holds.

Exercise 34:
Let $T \in \mathcal{L}(H)$ be self-adjoint. Show or calculate the following:
1. Let $\sigma(T) = \{0, 1\}$, then $T$ is orthogonal projector,
2. $\|TR(\lambda, T)^2\|_{\mathcal{L}(H)}$ for $\lambda \in \rho(T)$,
3. $\|Te^{-sT^2}\|_{\mathcal{L}(H)}$ for $s \geq 0$,
4. Assume that $T$ is positive $\|(I+sT)^{-1}e^{isT}\|_{\mathcal{L}(H)}$ for $s \geq 0$.

Exercise 35:
Let $A, B \in \mathcal{L}(H)$. Then

$$e^{A+B} = \lim_{N \to \infty} \left(e^{A/N}e^{B/N}\right)^N.$$