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Spectral Theory

11th Exercise Sheet

Exercise 36:
Let A be a self-adjoint operator in H. Let $f, g : \sigma(A) \to \mathbb{C}$ Borel measurable and locally bounded. Consider H_n, P_n and A_n be defined as in Thm 13.15. Let

$$f(A)x := \sum_{n\in\mathbb{N}} f(A_n)P_n x$$

for every $x \in \mathcal{D}(f(A))$ where $D(f(A)) = \{x \in X \,|\, \sum_{n\in\mathbb{N}} \|f(A_n)P_n x\|^2 < \infty\}$. Show that

1. $f(A) + g(A) \subseteq (f + g)(A)$,
2. $f(A)g(A) \subseteq (fg)(A)$.

Exercise 37:
Let $a \in \mathbb{R}^{d\times d}$ be symmetric such that $\xi^t a\xi \geq \eta\|\xi\|^2$ for certain $\eta > 0$ for every $\xi \in \mathbb{R}^d$ and $b \in \mathbb{R}^d$. We define

$$\mathfrak{a}(u, v) := \int \overline{\nabla v}^t a\nabla u + (b^t\nabla u)\overline{v}\,.$$

Show that

$$|\operatorname{Im}\mathfrak{a}(u, u)| \leq |b|\|\nabla u\|_2\|u\|_2 \leq \epsilon^2\|\nabla u\|_2^2 + \frac{|b|}{2\epsilon^2}\|u\|_2^2 \leq \frac{\epsilon^2}{4}\operatorname{Re}\mathfrak{a}(u, u) + \frac{|b|}{2\epsilon^2}\|u\|_2^2\,.$$

Notice that even for the case that a is real and symmetric it is possible that $|\operatorname{Im}\mathfrak{a}(u, u)| \neq 0$.

Exercise 38:
Find the adjoint operator and deficiency indices of it for the following cases:

1. $S = -\Delta$, $\mathcal{D}(S) := W^{2,2}(\mathbb{R})$,
2. $S = -i\frac{\partial}{\partial x}$, $\mathcal{D}(S) := W^{1,2}(\mathbb{R}^+)$,
3. $S = -i\frac{\partial}{\partial x}$, $\mathcal{D}(S) := W_0^{1,2}(\mathbb{R}^+)$.

Exercise 39:
Suppose D is a vector space. Let $s(f, g)$ be a sesquilinear form on D and $q(f) = s(f, f)$ the associated quadratic form. Prove the parallelogram law

$$q(f + g) + q(f - g) = 2q(f) + 2q(g)\,.$$

and the polarization identity

$$s(f, g) = \frac{1}{4}[q(f + g) - q(f - g)] + \frac{i}{4}[q(f - ig) - q(f + ig)]\,.$$

Show that $s(f, g)$ is symmetric if and only if $q(f)$ is real-valued.