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Spectral Theory

1st Exercise Sheet-Solutions

Exercise 1: (Prove the following claim)
Let X, Y be Banach spaces and $A : \mathcal{D}(A) \subseteq X \to Y$ be a closed operator. Furthermore assume that $r : [a, b] \to X$ is continuous with $r(t) \in \mathcal{D}(A)$ and $Ar : [a, b] \to Y$ is continuous for all $t \in [a, b]$ then

$$\int_a^b r(t)\mathrm{d}t \in \mathcal{D}(A) \text{ and } A\int_a^b r(t)\mathrm{d}t = \int_a^b Ar(t)\mathrm{d}t\,.$$

Proof
First we define for all $N \in \mathbb{N}$ the points

$$t_j^{(N)} := a + j\frac{b-a}{N}\,, \quad j \in 0, \dots, N\,.$$

Furthermore we set

$$x_N := \sum_{j=0}^{N} r\left(t_j^{(N)}\right)\frac{b-a}{N}\,.$$

It is easy to see that $x_N \in \mathcal{D}(A)$. It holds

$$\lim_{N\to\infty} x_N = \int_a^b r(t)\mathrm{d}t \quad \text{in}\, X$$

and also for every $N \in \mathbb{N}$

$$Ax_N = \sum_{j=0}^{N} Ar\left(t_j^{(N)}\right)\frac{b-a}{N}\,.$$

Using continuity of the mapping $t \mapsto Ar(t)$ we conclude

$$\lim_{N\to\infty} Ax_N = \int_a^b Ar(t)\mathrm{d}t \quad \text{in}\, Y.$$

Using the fact that A is closed concludes the proof.

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Exercise 2:
Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a domain, $\alpha \in \mathbb{N}_0^d$ be a multi-index and $1 \leq p, q \leq \infty$. Show that the operator of a weak derivative

$$\mathscr{D} = \frac{\partial^\alpha}{\partial x^\alpha} : \mathcal{D}(\mathscr{D}) := \{u \in L^p(\Omega) : \mathscr{D}u \in L^q(\Omega)\} \subseteq L^p(\Omega) \to L^q(\Omega)$$

is linear, densely defined and closed.

Proof
linear: It is obvious that the weak derivative is linear.
densely defined: We know that $C_c^\infty(\Omega) \subseteq \mathcal{D}(\mathscr{D})$. Furthermore $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for every $p \in [1, \infty)$.
closed: Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathscr{D})$, $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ such that $\lim_{n\to\infty} u_n = u$ in $L^p(\Omega)$ and $\lim_{n\to\infty} \mathscr{D}u_n = v$ in $L^q(\Omega)$. Now using definition we have

$$\begin{aligned}\int_\Omega u(x)\mathscr{D}\varphi(x)\mathrm{d}x &= \lim_{n\to\infty} \int_\Omega u_n(x)\mathscr{D}\varphi(x)\mathrm{d}x = \lim_{n\to\infty} (-1)^{|\alpha|} \int_\Omega \mathscr{D}u_n(x)\varphi(x)\mathrm{d}x \\ &= (-1)^{|\alpha|} \lim_{n\to\infty} \int_\Omega \mathscr{D}u_n(x)\varphi(x)\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v(x)\varphi(x)\mathrm{d}x\end{aligned}$$

for every $\varphi \in C_c^\infty(\Omega)$ which implies $u \in \mathcal{D}(\mathscr{D})$ and $v = \mathscr{D}u$.

Exercise 3:

Let X be a Banach space, $f : [a, b] \to X$ and $\Phi : [a, b] \to X'$. Show the following:

1. If $t \mapsto \langle f(t), \psi \rangle \in C^1[a, b]$ for all $\psi \in X'$ then $f \in C([a, b], X)$.
2. If $t \mapsto \langle g, \Phi(t) \rangle \in C^1[a, b]$ for all $g \in X$ then $\Phi \in C([a, b], X)$.
3. If $t \mapsto \langle f(t), \psi \rangle \in C^{k+1}[a, b]$ for all $\psi \in X'$ then $f \in C^k([a, b], X)$ for every $k \in \mathbb{N}$.
4. If $t \mapsto \langle g, \Phi(t) \rangle \in C^{k+1}[a, b]$ for all $g \in X$ then $\Phi \in C^k([a, b], X)$ for every $k \in \mathbb{N}$.

Proof

1. Using the uniform boundedness theorem there exists a constant N such that $\|f(t)\| \leq N$ for all $a \leq t \leq b$. Let $a \leq c \leq b$ then

$$\lim_{t \to 0} \langle t^{-1}[f(t+c) - f(c)], \psi \rangle = \frac{\mathrm{d}}{\mathrm{d}c} \langle f(c), \psi \rangle \,.$$

Using the uniform boundedness theorem we can find $M < \infty$ such that

$$\|t^{-1}[f(t+c) - f(c)]\| \leq M$$

for small enough t. This implies

$$\lim_{t \to 0} \|f(t+c) - f(c)\| = 0 \,.$$

2. This part in analogous to 1.
3. This part can be done using induction.
base case: The base case was already done in 1.
induction hypothesis: If $t \mapsto \langle f(t), \psi \rangle \in C^{l+1}[a, b]$ for all $\psi \in X'$ then $f \in C^l([a, b], X)$.
induction step: Let $t \mapsto \langle f(t), \psi \rangle \in C^{l+2}[a, b]$. By the uniform boundedness theorem we can find $g(t) \in X''$ for every $t \in [a, b]$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle f(t), \psi \rangle = \langle g(t), \psi \rangle \,.$$

Furthermore using induction hypothesis we have $g(t) \in C^l([a, b], X)$. This means that we are able to define

$$\int_0^t g(s) \mathrm{d}s$$

as an element of X'' and also

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle f(t) - f(a) - \int_a^t g(s) \mathrm{d}s, \psi \rangle = \frac{\mathrm{d}}{\mathrm{d}t} \langle f(t), \psi \rangle - \langle g(s), \psi \rangle$$

for all $t \in [a, b]$ and $\psi \in X'$. Then

$$f(t) - f(a) = \int_a^t g(s) \mathrm{d}s$$

for all $t \in [a, b]$ which implies

$$\lim_{h \to 0} h^{-1}[f(t+h) - f(t)] = \lim_{h \to 0} \int_t^{t+h} g(s) \mathrm{d}s = g(t) \,.$$

Thus $g(t) \in X$ and $f(t)$ is of class $C^{l+1}([a, b], X)$.
4. This part in analogous to 3.

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Exercise 4:
Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be a domain and $X := (C_b^0(\Omega), \|\cdot\|_\infty)$ be a set of all bounded continuous functions equipped with a supremum norm. For fixed $m \in X$ define

$$M_m : X \to X, f \mapsto m \cdot f \,.$$

Calculate the following:

1. spectrum of M_m and
2. resolvent function of M for every $\lambda \in \rho(M_m)$.

Proof
well defined: The product of two bounded, continuous functions is continuous and bounded.
linear: Product is linear.
continuous: We start by estimating $\|M_m\|_{L(X)}$. We show two inequalities. First

$$\|M_m f\|_X = \sup_{z\in\Omega} |m(z)f(z)| \leq \sup_{z\in\Omega} |m(z)| \sup_{z\in\Omega} |f(z)| \leq \|m\|_X \|f\|_X \,.$$

The latter holds for every $f \in X$ which gives $\|M_m\|_{L(X)} \leq \|m\|_X$. For the opposite inequality let $\tilde{f} = \chi_\Omega$. One has that $\tilde{f} \in X$ and $\|\tilde{f}\| = 1$. Then

$$\|M_m \tilde{f}\|_X = \sup_{z\in\Omega} |m(z)f(z)| = \|m\|_X \,.$$

This implies $\|M_m\|_{L(X)} = \|m\|_X$.
We show that $\sigma(M_m) = \overline{m(\Omega)}$.
$\subseteq$: Let $\lambda \in \mathbb{C} \setminus \overline{m(\Omega)}$. Then $z \mapsto \frac{1}{\lambda - m(z)}$ is bounded and $\frac{1}{\lambda - m(z)} \in X$. Then for every $f \in X$ we have

$$R_\lambda f := \frac{f}{\lambda - m} = M_{\frac{1}{\lambda - m(z)}} f \in X \,.$$

Obviously R_λ is linear on X. It is easy to check

$$R_\lambda(\lambda I_X - M_m) = (\lambda I_X - M_m) R_\lambda = I_X \,.$$

This implies $\lambda \in \rho(M_m)$ and gives corresponding resolvent as $R_\lambda = \frac{1}{\lambda - m}$.
$\supseteq$: Let $\lambda \in m(\Omega)$ then there exists $z_0 \in \Omega$ such that $m(z_0) = \lambda$. This implies

$$(\lambda I_X - M_m) f(z_0) = \lambda f(z_0) - m(z_0) f(z_0) = 0$$

for every $f \in X$, i.e.

$$\operatorname{ran}(\lambda I_X - M_m) \subseteq \{\in X : g(z_0) = 0\} \neq X \,.$$

This means that $\lambda I_X - M_m$ is not surjective, i.e. $\lambda \in \sigma(M_m)$. The spectrum is always closed and therefore $\overline{m(\Omega)} \subseteq \overline{\sigma(M_m)} = \sigma(M_m)$.

Exercise 5:
Let $X = (C[0,1], \|\cdot\|_\infty)$. For fixed $k \in C([0,1]\times[0,1])$ define the operator $T : X \to X$ as

$$(Tf)(t) := \int_0^t k(t,s)f(s)\mathrm{d}s\,.$$

for every $f \in X$ and $t \in [0,1]$. Show the following:

1. T is well defined,
2. $\|T\| \le \sup_{t\in[0,1]} \int_0^t |k(t,s)|\mathrm{d}s$ and
3. $\sigma(T) = \{0\}$.

Proof
Let $f \in X$, $\epsilon > 0$ and $\tilde{t} \in [0,1]$.
continuity of Tf: Without loss of generality we assume $f, k \neq 0$. The function k is uniformly continuous because it is continuous on a compact set, i.e. there exists $\delta > 0$ such that

$$|k(t_1,s_1) - k(t_2,s_2)| < \frac{\epsilon}{2\|f\|_X}$$

holds for all $t_1, t_2, s_1, s_2 \in [0,1]$ with

$$|t_1 - t_2| + |s_1 - s_2| < \delta\,.$$

Let $t \in [0,1]$ be

$$|t - \tilde{t}| < \max\left\{\delta, \frac{\epsilon}{2\|k\|_{C([0,1]\times[0,1])}\|f\|_X}\right\}$$

then

$$\begin{aligned}
|Tf(t) - Tf(\tilde{t})| &= \left|\int_0^t k(t,s)f(s)\mathrm{d}s - \int_0^{\tilde{t}} k(\tilde{t},s)f(s)\mathrm{d}s\right| \\
&= \left|\int_0^t k(t,s)f(s)\mathrm{d}s - \int_0^t k(\tilde{t},s)f(s)\mathrm{d}s + \int_0^t k(\tilde{t},s)f(s)\mathrm{d}s - \int_0^{\tilde{t}} k(\tilde{t},s)f(s)\mathrm{d}s\right| \\
&= \left|\int_0^t [k(t,s) - k(\tilde{t},s)]f(s)\mathrm{d}s - \int_t^{\tilde{t}} k(\tilde{t},s)f(s)\mathrm{d}s\right| \\
&\le \int_0^t \left|k(t,s) - k(\tilde{t},s)\right| |f(s)|\mathrm{d}s + \int_t^{\tilde{t}} \left|k(\tilde{t},s)\right| |f(s)|\mathrm{d}s \\
&\le \int_0^t \frac{\epsilon}{2\|f\|_X}\|f\|_X\mathrm{d}s + \left|\int_t^{\tilde{t}} \|k\|_{C([0,1]\times[0,1])}\|f\|_X\mathrm{d}s\right| \\
&\le \frac{\epsilon}{2}t + \|k\|_{C([0,1]\times[0,1])}\|f\|_X|t - \tilde{t}| \\
&\le \frac{\epsilon}{2} + \|k\|_{C([0,1]\times[0,1])}\|f\|_X\frac{\epsilon}{2\|k\|_{C([0,1]\times[0,1])}\|f\|_X} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

which shows that Tf is continuous on $[0,1]$.
T *linear*: T is linear, because the integral is linear.

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continuity of T: It holds

$$\begin{aligned}\|Tf\|_{L(X)} &= \sup_{t\in[0,1]}\left|\int_0^t k(t,s)f(s)\mathrm{d}s\right| \\ &= \sup_{t\in[0,1]}\int_0^t |k(t,s)|\,|f(s)|\mathrm{d}s \\ &= \sup_{t\in[0,1]}\int_0^t |k(t,s)|\,\mathrm{d}s\|f(s)\|_X\end{aligned}$$

i.e. T is continuous on X.
$\sigma(T)=\{0\}$: We start by showing that $r(T)=0$. Using induction we show:

$$|T^n f(t)| \le \frac{1}{n!}t^n\|k\|^n_{C([0,1]\times[0,1])}\|f\|_X\,.$$

for all $n\in\mathbb{N}$ and $t\in[0,1]$.
base case: It holds

$$|Tf(t)| = \left|\int_0^t k(t,s)f(s)\mathrm{d}s\right| \le \int_0^t \|k\|_{C([0,1]\times[0,1])}\|f\|_X\mathrm{d}s \le t\|k\|_{C([0,1]\times[0,1])}\|f\|_X\,.$$

induction hypothesis: Let for fixed $n\in\mathbb{N}$ and for all $s\in[0,1]$ holds

$$|T^n f(s)| \le \frac{1}{n!}s^n\|k\|^n_{C([0,1]\times[0,1])}\|f\|_X\,.$$

induction step: Using induction step we have

$$\begin{aligned}|T^{n+1}f(t)| = |T(T^n f)(t)| &= \left|\int_0^t k(t,s)T^n f(s)\mathrm{d}s\right| \\ &\le \int_0^t |k(t,s)|\,|T^n f(s)|\mathrm{d}s \\ &\le \int_0^t \|k\|_{C([0,1]\times[0,1])}\frac{1}{n!}s^n\|k\|^n_{C([0,1]\times[0,1])}\|f\|_X\mathrm{d}s \\ &= \frac{1}{n!}\|k\|^{n+1}_{C([0,1]\times[0,1])}\|f\|_X\int_0^t s^n\mathrm{d}s \\ &= \frac{1}{(n+1)!}t^{n+1}\|k\|^{n+1}_{C([0,1]\times[0,1])}\|f\|_X\end{aligned}$$

Estimating t we obtain

$$\|T^n f\|_X \le \frac{1}{n!}\|k\|^n_{C([0,1]\times[0,1])}\|f\|_X$$

and

$$\|T^n\|_{L(X)} \le \frac{1}{n!}\|k\|^n_{C([0,1]\times[0,1])}$$

for all $n\in\mathbb{N}$. This implies

$$r(T) = \lim_{n\to\infty}\|T^n\|^{\frac{1}{n}}_{L(X)} \le \frac{\|k\|_{C([0,1]\times[0,1])}}{\sqrt[n]{n!}} = 0$$

where we used Stirling formula in the last step. Due to the fact that the spectrum of bounded operator is always non-empty we obtain $\sigma(T)=\{0\}$.