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Spectral Theory

2nd Exercise Sheet-Solutions

Exercise 6:
Let A be a closed linear operator in X.

1. Let X be reflexive, $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(A)$ such that $x_n \to x$ in X and $\sup_n \|Ax_n\| < \infty$. Show that $x \in \mathcal{D}(A)$.
2. This is false, in general, if X is not reflexive. Show this for the following:
   Let $X = C([0,1])$ and $A = \frac{\mathrm{d}}{\mathrm{d}x}$ with the domain $\mathcal{D}(A) = C^1([0,1])$.

Proof

1. $\Rightarrow$: Using assumptions we have that Ax_n is a bounded sequence. The reflexivity of X implies the existence of a subsequence x_{k_n} s.t. $Ax_{k_n} \rightharpoonup y$. This means that the operator A is weakly closed. From FA-Ex. 57 we know

   $$A \text{ closed} \Leftrightarrow A \text{ weakly closed}\,.$$

   $\Leftarrow$: A is closed implies that A is weakly closed which implies that $\mathrm{gr}(A)$ is weakly closed. Using Thm. 6.13 completes the proof.

2. We find a sequence s.t. its limit is in C but not in C^1 with bounded derivative, e.g. $f_n = \left|1 - \frac{1}{2}\right|^{1+\frac{1}{n}}$.

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Exercise 7:
Let A be a linear operator from X to Y. Show that the following are equivalent:

1. The operator A is closable,
2. $\overline{\mathrm{gr}(A)}$ is a graph of a linear operator from X to Y,
3. if x_n is a sequence in $\mathcal{D}(A)$ such that $x_n \to 0$ in X and $Ax_n \to y$ in Y then $y = 0$.

Proof
$1 \Rightarrow 2$: Due to the assumption that A is closable there exists $\overline{A}$ a closed extension. Let $(x, y) \in \overline{\mathrm{gr}(A)}$ and choose a sequence $(x_n, y_n) \in \mathrm{gr}(A)$ s.t. $(x_n, y_n) \to (x, y)$. Then

$$\lim_{n\to\infty} \overline{A}x_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} y_n = y \in Y$$

along with $\lim x_n = x$ and the fact that $\overline{A}$ completes the proof.

$2 \Rightarrow 3$: Let B be an operator $\mathrm{gr}(B) = \overline{\mathrm{gr}(A)}$. B is linear with a closed graph, i.e. B is closed. Let $y \in Y$, $(x_n) \subseteq \mathcal{D}(A)$ s.t. $\lim_{n\to\infty} x_n = 0$ in X and $\lim_{n\to\infty} Ax_n = y$ in Y. We have

$$y = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n$$

Closedness of B leads to $y = 0$.

$2 \Rightarrow 1$: Let B be an operator $\mathrm{gr}(B) = \overline{\mathrm{gr}(A)}$. B is linear with a closed graph, i.e. B is closed. Also $x \in \mathcal{D}(A)$ satisfies $Bx = Ax$.

$3 \Rightarrow 2$: We prove this by contradiction i.e. $\neg 2 \wedge 3$ can not hold. Assume that $\overline{\mathrm{gr}(A)}$ is not a graph of an operator. Then there exists $x \in X$ and $y_1, y_2 \in Y$ s.t. $(x, y_1), (x_2, y) \in \overline{\mathrm{gr}(A)}$, $y_1 \neq y_2$. We choose two sequences $(x_n), (z_n) \in \mathcal{D}(A)$ s.t. $\lim_{n\to\infty} x_n = x = \lim_{n\to\infty} z_n$ in X and $\lim_{n\to\infty} Ax_n = y_1$, $\lim_{n\to\infty} Az_n = y_2$. We define

$$y := y_1 - y_2 = \lim_{n\to\infty} Ax_n - \lim_{n\to\infty} Az_n = \lim_{n\to\infty} (Ax_n - Az_n) = \lim_{n\to\infty} A(x_n - z_n) \in Y$$

and $\lim_{n\to\infty}(x_n - z_n) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} z_n = x - x = 0$ in X. Using 2) we get $y_1 - y_2 = y = 0$ which is a contradiction.

Exercise 8:
Let X be a Banach space and $\Phi : \mathcal{D}(\Phi) \subseteq X \to \mathbb{C}$ be linear and $\mathcal{D}(\Phi)$ be dense in X. Then

$$\Phi \text{ closable } \Leftrightarrow \Phi \text{ bounded}\,.$$

Proof
Φ bounded $\Rightarrow$ Φ closable: Φ is bounded which means that we can extend Φ by continuity

$$\|\tilde{\Phi}(x_n - x)\| \leq \|\Phi\|\|x_n - x\| \Rightarrow \Phi(x_n) \to \Phi(x)\,.$$

We have $\Phi(x_n) \to \Phi(x)$ and $\Phi(x_n) \to c$ which implies $c = \Phi(x)$ because $\mathbb{C}$ is Hausdorf.

Φ closable $\Rightarrow$ Φ bounded: We show this by contraposition, i.e. Φ not bounded $\Rightarrow$ Φ not closable. Due to the assumtion that Φ is not bounded we know that there exists (x_n), $\|x_n\| = 1$ s.t. $\|\Phi(x_n)\| \to \infty$. However this is equivalent to existence of sequence $z_n \to 0$ s.t. $\|\Phi(z_n)\| = 1$. Due to the compactness of $\mathbb{C}$ there exists converging subsequence of $\Phi(x_n)$ which implies that Φ is not closable by Ex. 7)-3.

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Exercise 9:

1. Let $m \in L^\infty(\mathbb{R})$ be fixed. We define the operator $(Sf)(t) := m(t)f(t+1)$ for $f \in L^1(\mathbb{R})$ and $t \in \mathbb{R}$ a.e. Show that the operator S is bounded and determine $S' \in \mathcal{L}(L^\infty(\mathbb{R}))$.
2. Let $T : L^1(0,1) \to c_0$ be defined as $(Tf)_n = \int_0^1 f(t)t^n \mathrm{d}t$. Show that the operator T is linear, bounded and determine T'.
3. Let $X = L^2(0,1)$ and $A = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with the domain $\mathcal{D}(A) = C_c^\infty(0,1)$. Show that $W^{2,2}(0,1) \subseteq \mathcal{D}(A')$, $A'g = g''$ for $g \in W^{2,2}(0,1)$ and that A is closable.

Proof

1. It is obvious that S is linear because integral is linear. We write

$$\|Sf\|_1 = \int_{\mathbb{R}} |m(t)f(t+1)|\mathrm{d}t \leq \|m\|_\infty \|f\|_1$$

which shows that S is bounded. This means that $S \in \mathcal{L}(X)$ because it it bounded and linear. Let $g \in L^\infty(\mathbb{R})$ then

$$\langle Sf, g\rangle = \int_{\mathbb{R}} m(t)f(t+1)g(t)\mathrm{d}t = \int_{\mathbb{R}} f(t)m(t-1)g(t-1)\mathrm{d}t$$

which means $S'g(t) = m(t-1)g(t-1)$.

2. For $t \in (0,1)$ we have $|t^m f(t)| \leq |f(t)|$. We have

$$\lim_{n\to\infty} Tf = \lim_{n\to\infty} \left(\int_0^1 t^n f(t)\right) = \int_0^1 \chi_1(t)f(t) = 0$$

which means $Tf \in c_0$, i.e. T is well defined. T is linear because integral is linear. T is bounded because

$$\|Tf\|_{c_0} = \sup \left|\int_0^1 t^n f(t)\mathrm{d}t\right| \leq \int_0^1 |f(t)|\mathrm{d}t = \|f\|_1 \,.$$

We have $\|T\| \leq 1$, $f \in L^1$. Let $a \in l^1 \approx c_0'$

$$\langle Tf, a\rangle = \sum_{n=1}^\infty \int_0^1 t^n f(t)\mathrm{d}t a_n = \sum_{n=1}^\infty \int_0^1 a_n t^n f(t)\mathrm{d}t = \int_0^1 f(t)\sum_{n=1}^\infty a_n a_n t^n \mathrm{d}t$$

i.e.

$$\langle Tf, a\rangle = \left\langle f, \sum a_n t^n \right\rangle_{L^1((0,1))\times L^\infty((0,1))} .$$

We conclude $T' \in \mathcal{L}(l^1(\mathbb{N},\mathbb{R}), L^\infty((0,1)))$, $T'a = \sum_{n=1}^\infty a_n t^n$, $a = a_n \in l^1$.

3. We have

$$\langle A\varphi, f\rangle = \int \varphi'' f \mathrm{d}x = \int \varphi g \mathrm{d}x$$

where we write $g = A'f = f''$ for $g \in W^{2,2}((0,1))$. this means $W^{2,2}((0,1)) \subseteq \mathcal{D}(A')$. A is closable because for every $f_n \to 0$, $f_n \in C_c^\infty((0,1))$ we have $f_n'' \to 0$ and $f_n'' \in \mathbb{C}_c^\infty((0,1))$.