Exercise 6:
Let $A$ be a closed linear operator in $X$.

1. Let $X$ be reflexive, $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{D}(A)$ such that $x_n \to x$ in $X$ and $\sup_n \|Ax_n\| < \infty$. Show that $x \in \mathcal{D}(A)$.

2. This is false, in general, if $X$ is not reflexive. Show this for the following:
   Let $X = C([0,1])$ and $A = \frac{d}{dx}$ with the domain $\mathcal{D}(A) = C^1([0,1])$.

Proof

1. $\Rightarrow$: Using assumptions we have that $Ax_n$ is a bounded sequence. The reflexivity of $X$ implies the existence of a subsequence $x_{k_n}$ s.t. $Ax_{k_n} \rightharpoonup y$. This means that the operator $A$ is weakly closed. From FA-Ex. 57 we know

   $A$ closed $\iff$ $A$ weakly closed.

   $\Leftarrow$: $A$ is closed implies that $A$ is weakly closed which implies that $\text{gr}(A)$ is weakly closed. Using Thm. 6.13 completes the proof.

2. We find a sequence s.t. its limit is in $C$ but not in $C^1$ with bounded derivative, e.g. $f_n = |1 - \frac{1}{2^n}|^{1 + \frac{1}{n}}$. 

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Exercise 7:
Let $A$ be a linear operator from $X$ to $Y$. Show that the following are equivalent:

1. The operator $A$ is closable,
2. $\text{gr}(A)$ is a graph of a linear operator from $X$ to $Y$,
3. if $x_n$ is a sequence in $\mathcal{D}(A)$ such that $x_n \to 0$ in $X$ and $Ax_n \to y$ in $Y$ then $y = 0$.

Proof
1 $\Rightarrow$ 2: Due to the assumption that $A$ is closable there exists $\overline{A}$ a closed extension. Let $(x, y) \in \text{gr}(A)$ and choose a sequence $(x_n, y_n) \in \text{gr}(A)$ s.t. $(x_n, y_n) \to (x, y)$. Then

$$\lim_{n \to \infty} \overline{A} x_n = \lim_{n \to \infty} A x_n = \lim_{n \to \infty} y_n = y \in Y$$

along with $\lim x_n = x$ and the fact that $\overline{A}$ completes the proof.

2 $\Rightarrow$ 3: Let $B$ be an operator $\text{gr}(B) = \text{gr}(A)$. $B$ is linear with a closed graph, i.e. $B$ is closed. Let $y \in Y$, $(x_n) \subseteq \mathcal{D}(A)$ s.t. $\lim_{n \to \infty} x_n = 0$ in $X$ and $\lim_{n \to \infty} Ax_n = y$ in $Y$. We have

$$y = \lim_{n \to \infty} A x_n = \lim_{n \to \infty} B x_n$$

Closedness of $B$ leads to $y = 0$.

2 $\Rightarrow$ 1: Let $B$ be an operator $\text{gr}(B) = \text{gr}(A)$. $B$ is linear with a closed graph, i.e. $B$ is closed. Also $x \in \mathcal{D}(A)$ satisfies $B x = A x$.

3 $\Rightarrow$ 2: We prove this by contradiction i.e. $\neg 2 \land 3$ can not hold. Assume that $\overline{\text{gr}(A)}$ is not a graph of an operator. Then there exists $x \in X$ and $y_1, y_2 \in Y$ s.t. $(x, y_1), (x_2, y) \in \text{gr}(A)$, $y_1 \neq y_2$. We choose two sequences $(x_n), (z_n) \in \mathcal{D}(A)$ s.t. $\lim_{n \to \infty} x_n = x = \lim_{n \to \infty} z_n$ in $X$ and $\lim_{n \to \infty} Ax_n = y_1$, $\lim_{n \to \infty} Az_n = y_2$. We define

$$y := y_1 - y_2 = \lim_{n \to \infty} A x_n - \lim_{n \to \infty} A z_n = \lim_{n \to \infty} (Ax_n - Az_n) = \lim_{n \to \infty} A (x_n - z_n) \in Y$$

and $\lim_{n \to \infty} (x_n - z_n) = \lim_{n \to \infty} x_n - \lim_{n \to \infty} z_n = x - x = 0$ in $X$. Using 2) we get $y_1 - y_2 = y = 0$ which is a contradiction.
Exercise 8:
Let $X$ be a Banach space and $\Phi : \mathcal{D}(\Phi) \subseteq X \to \mathbb{C}$ be linear and $\mathcal{D}(\Phi)$ be dense in $X$. Then

$$\Phi \text{ closable } \iff \Phi \text{ bounded}.$$ 

Proof

$\Phi$ bounded $\Rightarrow$ $\Phi$ closable: $\Phi$ is bounded which means that we can extend $\Phi$ by continuity

$$\|\Phi(x_n - x)\| \leq \|\Phi\|\|x_n - x\| \Rightarrow \Phi(x_n) \to \Phi(x).$$

We have $\Phi(x_n) \to \Phi(x)$ and $\Phi(x_n) \to c$ which implies $c = \Phi(x)$ because $\mathbb{C}$ is Hausdorff.

$\Phi$ closable $\Rightarrow$ $\Phi$ bounded: We show this by contraposition, i.e. $\Phi$ not bounded $\Rightarrow$ $\Phi$ not closable. Due to the assumption that $\Phi$ is not bounded we know that there exists $(x_n), \|x_n\| = 1$ s.t. $\|\Phi(x_n)\| \to \infty$. However this is equivalent to existence of sequence $z_n \to 0$ s.t. $\|\Phi(z_n)\| = 1$. Due to the compactness of $\mathbb{C}$ there exists converging subsequence of $\Phi(x_n)$ which implies that $\Phi$ is not closable by Ex. 7)-3.
Exercise 9:
1. Let \( m \in L^\infty(\mathbb{R}) \) be fixed. We define the operator \((Sf)(t) := m(t)f(t + 1)\) for \( f \in L^1(\mathbb{R}) \) and \( t \in \mathbb{R} \) a.e. Show that the operator \( S \) is bounded and determine \( S' \in \mathcal{L}(L^\infty(\mathbb{R})) \).
2. Let \( T : L^1(0,1) \to c_0 \) be defined as \((Tf)_n = \int_0^1 f(t)t^n dt\). Show that the operator \( T \) is linear, bounded and determine \( T' \).
3. Let \( X = L^2(0,1) \) and \( A = \frac{d^2}{dx^2} \) with the domain \( \mathcal{D}(A) = C_c^\infty(0,1) \). Show that \( W^{2,2}(0,1) \subseteq \mathcal{D}(A') \), \( A'g = g'' \) for \( g \in W^{2,2}(0,1) \) and that \( A \) is closable.

Proof

1. It is obvious that \( S \) is linear because integral is linear. We write
\[
\|Sf\|_1 = \int_\mathbb{R} |m(t)f(t + 1)|dt \leq \|m\|_\infty \|f\|_1
\]
which shows that \( S \) is bounded. This means that \( S \in \mathcal{L}(X) \) because it is bounded and linear. Let \( g \in L^\infty(\mathbb{R}) \) then
\[
\langle Sf, g \rangle = \int_\mathbb{R} m(t)f(t + 1)g(t)dt = \int_\mathbb{R} f(t)m(t - 1)g(t - 1)dt
\]
which means \( S'g(t) = m(t - 1)g(t - 1) \).

2. For \( t \in (0,1) \) we have \(|t^m f(t)| \leq |f(t)|\). We have
\[
\lim_{n \to \infty} Tf = \lim_{n \to \infty} \left( \int_0^1 t^n f(t) \right) = \int_0^1 \chi_1(t)f(t) = 0
\]
which means \( Tf \in c_0 \), i.e. \( T \) is well defined. \( T \) is linear because integral is linear. \( T \) is bounded because
\[
\|Tf\|_{c_0} = \sup \left| \int_0^1 t^n f(t)dt \right| \leq \int_0^1 |f(t)|dt = \|f\|_1.
\]
We have \( \|T\| \leq 1 \), \( f \in L^1 \). Let \( a \in l^1 \approx c_0^d \)
\[
\langle Tf, a \rangle = \sum_{n=1}^\infty \int_0^1 \int_0^1 t^n f(t)dt a_n = \sum_{n=1}^\infty \int_0^1 a_n t^n f(t)dt = \int_0^1 f(t) \sum_{n=1}^\infty a_n a_n t^n dt
\]
i.e.
\[
\langle Tf, a \rangle = \left< f, \sum a_n t^n \right>_{L^1((0,1)) \times L^\infty((0,1))}.
\]
We conclude \( T' \in \mathcal{L}(l^1(N, \mathbb{R}), L^\infty((0,1))) \), \( T'a = \sum_{n=1}^\infty a_n t^n \), \( a = a_n \in l^1 \).

3. We have
\[
\langle A\varphi, f \rangle = \int \varphi''f dx = \int \varphi g dx
\]
where we write \( g = A'f = f'' \) for \( g \in W^{2,2}((0,1)) \). This means \( W^{2,2}((0,1)) \subseteq \mathcal{D}(A') \). \( A \) is closable because for every \( f_n \to 0 \), \( f_n \in C_c^\infty((0,1)) \) we have \( f''_n \to 0 \) and \( f''_n \in C_c^\infty((0,1)) \).

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