Spectral Theory

3rd Exercise Sheet - Solutions

Exercise 10:
Let $A$ be a linear operator in $X$ and let $X_0 \subseteq X$ be a closed linear subspace invariant under the resolvents of $A$. Define $A_0 := A|_{D(A_0)}$ where $D(A_0) = \{ x \in X_0 \cap D(A), Ax \in X_0 \}$. Show that $\rho(A) \subseteq \rho(A_0)$.

Proof
We show that $\lambda \in \rho(A)$ implies and $\lambda \in \rho(A_0)$ and $R(\lambda, A_0) = R(\lambda, A)|_{X_0}$. Let $A_0 := A|_{D(A_0)}$.

Let $\lambda \in \rho(A)$ then

$$\lambda - A : D(A) \to X \text{ injective}$$

and also

$$\lambda - A_0 : D(A_0) \to X \text{ injective}$$

because $A_0$ is restriction of $A$. Now we check that $\lambda - A : D(A) \to X$ is surjective. Let $y \in X_0$. We set $x := R(\lambda, A)y$. Then $x \in D(A)$. Using assumptions we have $x \in X_0$ which implies $x \in D(A) \cap X$. Furthermore we have

$$(\lambda - A)x = (\lambda - A)R(\lambda, A)y = y \in X_0.$$ 

This gives

$$Ax = \lambda x - y \in X_0,$$

which implies $x \in D(A_0)$ and $(\lambda - A_0)x = y$ i.e. $\lambda - A_0$ is surjective.
Exercise 11:
Let \( X = C([0,1]) \), \( m \in X \) and \( T : X \to X \) defined by \( Tf = mf \). Determine \( \sigma_p(T) \), \( \sigma_c(T) \) and \( \sigma_r(T) \).

Proof
From Ex. 4 we already know that \( \sigma(M_m) = \overline{m([0,1])} = m([0,1]) \). Let \( \lambda_0 \in m([0,1]) \) then there exists \( t_0 \in [0,1] \) s.t. \( m(t_0) = \lambda_0 \). Let \( g \in R(\lambda_0 - M_m) \) then there exists \( f \in X \) s.t.

\[
g(t_0) = (\lambda_0 - M_m)f(t_0) = (\lambda_0 - m(t_0))f(t_0) = 0.
\]

In other words \( R(\lambda_0 - M_m) \subseteq \{ g \in X : g(t_0) = 0 \} \neq X \). This means that \( \sigma_r(M_m) = R(m) \) and \( \sigma_c(M_m) = \emptyset \).

Let \( \lambda_0 \in \sigma_p(M_m) \) then there exists \( f_0 \in X \setminus \{0\} \) s.t.

\[
mf_0 = M_mf_0 = \lambda_0f_0.
\]

Function \( f_0 \) is continuous and not identically equal to 0, i.e. there must be \( 0 \leq a \leq b \leq 1 \) s.t. \( f_0(x) \neq 0 \) for all \( x \in [a,b] \). This implies \( m(x) \overset{1}{=} \lambda_0 \) for all \( x \in [a,b] \). We introduce

\[
P(m) := \{ \lambda \in \mathbb{C} : 0 \leq \alpha < \beta \leq 1 \text{ s.t. } m(x) = \lambda \forall x \in [\alpha,\beta] \}\]

Now we construct eigenfunction \( f_0 \in X \) to eigenvalue \( \lambda_0 \in P(m) \) as

\[
f_0(x) := \begin{cases} 1 & x \in [\alpha,\beta] \\ 0 & x \notin [a,b] \\ [0,1] & \text{otherwise} \end{cases}
\]

where we have \( 0 \leq a < \alpha < \beta < b \leq 1 \) s.t. \( m(x) = \lambda_0 \) for all \( x \in [a,b] \). We write

\[
M_m f_0(x) = m(x)f_0(x) = \lambda_0f_0(x)
\]

which holds for all \( x \in [0,1] \). We conclude \( \sigma_p(M_m) = P(m) \).
Exercise 12:
Show that the Fourier transform $F: \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is well defined automorphism.

Proof
First we prove the following Lemma.

Lemma: Let $(f_n)_n \subseteq \mathcal{S}(\mathbb{R}^d)$ be a sequence s.t. $\lim_{n \to \infty} f_n = f$ in $\mathcal{S}(\mathbb{R}^d)$. Then for all $p \in [1, \infty]$, $\beta \in \mathbb{N}^d$ $\lim_{n \to \infty} f_n = f$ in $L^p(\mathbb{R}^d)$ and

$$\|\partial^\beta g\|_{L^p(\mathbb{R}^d)} \leq C_{p,d} \sum_{\alpha \in \mathbb{N}^d:|\alpha| \leq \left\lfloor \frac{d+1}{p} \right\rfloor} \rho_{\alpha,\beta}(g)$$

for all $g \in \mathcal{S}(\mathbb{R}^d)$ and given $C_{p,d} > 0$ where $\rho_{\alpha,\beta}(g) := \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta g(x)|$.

Lemma-proof:
First we show that $\forall x \in \mathbb{R}^d$ and $k \in \mathbb{N}_0$:

$$|x|^k \leq C_{d,k} \sum_{\alpha \in \mathbb{N}^d:|\alpha| = k} |x|^\alpha.$$

We consider a function $h : \mathbb{R}^D \to [0, \infty) : x \mapsto \sum_{\alpha \in \mathbb{N}^d:|\alpha| = k} |x|^\alpha$. Obviously $h$ is continuous. This implies that $h(S^{d-1})$ is compact because a surface of a unit ball in $\mathbb{R}^d$ is compact. The set $h(S^{d-1})$ has maximum and minimum. We denote a minimum by $\kappa_{k,d} := \min h(S^{d-1}) \in (0, \infty)$. The minimum is bigger then zero because $h > 0$ on $S^{d-1}$ which follows from $h(x) = 0 \iff x = 0$. For arbitrary $x \in \mathbb{R}^d$ we have

$$\sum_{\alpha \in \mathbb{N}^d:|\alpha| = k} |x|^\alpha = \sum_{\alpha \in \mathbb{N}^d:|\alpha| = k} \left| \frac{x}{|x|} \right|^\alpha |x|^k \sum_{\alpha \in \mathbb{N}^d:|\alpha| = k} \left| \frac{x}{|x|} \right|^\alpha \geq \kappa_{k,d} |x|^k$$

which is equivalent to

$$|x|^k \leq \frac{1}{\kappa_{k,d}} \sum_{\alpha \in \mathbb{N}^d:|\alpha| = k} |x|^\alpha.$$

The case for $x = 0$ is obvious.

Case $p = \infty$:
Then for $g \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\|\partial^\beta g\|_{L^\infty(\mathbb{R}^d)} = \rho_{0,\beta}(g) \leq \sum_{\alpha \in \mathbb{N}^d:|\alpha| \leq 1} \rho_{\alpha,\beta}(g)$$

i.e. $C_{\infty,d} = 1 > 0$.

Case $p \in [0, \infty)$:
We write

$$\|\partial^\beta g\|_{L^p(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} |\partial^\beta g(x)|^p dx \right)^{\frac{1}{p}} \leq \left( \int_{|x| < 1} |\partial^\beta g(x)|^p dx + \int_{|x| > 1} |\partial^\beta g(x)|^p dx \right)^{\frac{1}{p}} \leq \left( \int_{|x| < 1} \|\partial^\beta g(x)\|^p_{L^p} dx + \int_{|x| > 1} |x|^{d+1} |x|^{-d+1} |\partial^\beta g(x)|^p dx \right)^{\frac{1}{p}} \leq \left( L^d(B_1(0)) \frac{\rho_{\infty,p}(g)^p}{\rho_p} + \sup_{|x| > 1} (\frac{|x|^{d+1} |\partial^\beta g(x)|}{\rho_{\infty,p}(g)}) \right)^{\frac{1}{p}} \leq \left( L^d(B_1(0)) \int_{|x| > 1} |x|^{-d+1} dx \right)^{\frac{1}{p}} \left( \|\partial^\beta g(x)\|_{L^\infty} + \sup_{x \in \mathbb{R}^d} (\int_{x \in \mathbb{R}^d} |x|^{\frac{d+1}{p}} |\partial^\beta g(x)|) \right)$$
where we used \((a + b)^{\frac{1}{p}} \leq a^{\frac{1}{p}} + b^{\frac{1}{p}}\) for \(a, b > 0\) and \(p \geq 1\). Now using inequality shown before we get

\[
\|\partial^\beta g\|_{L^p(\mathbb{R}^d)} \leq \max \left\{ \mathcal{L}^d(B_1(0)), \int_{|x| > 1} |x|^{-d+1} \, dx \right\} \left( \|\partial^\beta g(x)\|_\infty + \sup_{x \in \mathbb{R}^d} C_d, \left[ \frac{d+1}{p} \right] \sum_{\alpha \in \mathbb{N}_0^d: |\alpha| = \left[ \frac{d+1}{p} \right]} |x|^\alpha |\partial^\beta g(x)| \right) \leq \max \left\{ \mathcal{L}^d(B_1(0)), \int_{|x| > 1} |x|^{-d+1} \, dx \right\} \left( \rho_{\alpha, \beta}(g) + C_d, \left[ \frac{d+1}{p} \right] \sum_{\alpha \in \mathbb{N}_0^d: |\alpha| = \left[ \frac{d+1}{p} \right]} \rho_{\alpha, \beta}(g) \right) \leq C_{p,d} \sum_{\alpha \in \mathbb{N}_0^d: |\alpha| \leq \left[ \frac{d+1}{p} \right]} \rho_{\alpha, \beta}(g) \]

\[
\square
\]

The Lemma above can be used to show that if a sequence \((f_n) \in \mathcal{S}(\mathbb{R}^d)\) converges in \(\mathcal{S}(\mathbb{R}^d)\) it also converges in \(L^p(\mathbb{R}^d)\) because

\[
\|f_n - f\|_{L^p(\mathbb{R}^d)} \leq C_{p,d} \sum_{\alpha \in \mathbb{N}_0^d: |\alpha| \leq \left[ \frac{d+1}{p} \right]} \rho_{\alpha, \beta}(f_n - f).
\]

Now we are ready to show that \(\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)\) is well defined automorphism.

Well defined: Consider \(f \in \mathcal{S}(\mathbb{R}^d)\) then

\[
\|\partial^\beta f\|_\infty = \left( \frac{2\pi}{{\pi}} \right)^{|\beta|} \frac{|\beta|}{|\alpha|} \|\partial^\alpha \partial^\beta f\|_\infty \leq \frac{(2\pi)^{|\beta|}}{(2\pi)^{|\alpha|}} \|\partial^\alpha \partial^\beta f\|_1.
\]

We know that \(f \in \mathcal{S}(\mathbb{R}^d)\) implies \(x^\alpha \partial^\beta f \in \mathcal{S}(\mathbb{R}^d)\) which gives \(\hat{f} \in \mathcal{S}(\mathbb{R})\).

Linear: The integral is linear.

Continuous: Let \(f_n \rightarrow f\) in \(\mathcal{S}(\mathbb{R}^d)\) then \(\forall \alpha, \beta \in \mathbb{N}_0^d:\)

\[
\rho_{\alpha, \beta}(\hat{f}_n - \hat{f}) \leq \frac{(2\pi)^{|\beta|}}{(2\pi)^{|\alpha|}} \|\partial^\alpha (\partial^\beta (f_n - f))\|_1 \leq C_{1,d} \frac{(2\pi)^{|\beta|}}{(2\pi)^{|\alpha|}} \sum_{\hat{\alpha} \in \mathbb{N}_0^d: \hat{|\alpha|} \leq d+2} \rho_{\hat{\alpha}, \alpha}((\partial^\beta (f_n - f)) \leq C_{1,d} \frac{(2\pi)^{|\beta|}}{(2\pi)^{|\alpha|}} \sum_{\hat{\alpha} \in \mathbb{N}_0^d: \hat{|\alpha|} \leq d+2} \rho_{\hat{\alpha} + \beta, \alpha}(f_n - f)
\]

which converges to 0 as \(n \rightarrow \infty\) because on the right hand side we have a finite sum.

Bijective: Consider inverse in the form \(\mathcal{F}^{-1}(f)(x) = \hat{f}(-x)\). From lecture we have

\[
\mathcal{F}^{-1} \mathcal{F} = \mathcal{F} \mathcal{F}^{-1} = f
\]

for all \(f \in \mathcal{S}(\mathbb{R}^d)\) and every \(x \in \mathbb{R}^d\). This shows that \(\mathcal{F}\) is well defined automorphism on \(\mathcal{S}(\mathbb{R}^d)\).