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Spectral Theory

3rd Exercise Sheet - Solutions

Exercise 10:
Let A be a linear operator in X and let $X_0 \subseteq X$ be a closed linear subspace invariant under the resolvents of A. Define $A_0 := A|_{\mathcal{D}(A_0)}$ where $\mathcal{D}(A_0) = \{x \in X_0 \cap \mathcal{D}(A), Ax \in X_0\}$. Show that $\rho(A) \subseteq \rho(A_0)$.

Proof
We show that $\lambda \in \rho(A)$ implies and $\lambda \in \rho(A_0)$ and $R(\lambda, A_0) = R(\lambda, A)|_{X_0}$. Let $A_0 := A|_{\mathcal{D}(A_0)}$. Let $\lambda \in \rho(A)$ then

$$\lambda - A : \mathcal{D}(A) \to X \text{ injective}$$

and also

$$\lambda - A_0 : \mathcal{D}(A_0) \to X \text{ injective}$$

because A_0 is restriction of A. Now we check that $\lambda - A : \mathcal{D}(A) \to X$ is surjective. Let $y \in X_0$. We set $x := R(\lambda, A)y$. Then $x \in \mathcal{D}(A)$. Using assumptions we have $x \in X_0$ which implies $x \in \mathcal{D}(A) \cap X$. Furthermore we have

$$(\lambda - A)x = (\lambda - A)R(\lambda, A)y = y \in X_0\,.$$

This gives

$$Ax = \lambda x - y \in X_0\,.$$

which implies $x \in \mathcal{D}(A_0)$ and $(\lambda - A_0)x = y$ i.e. $\lambda - A_0$ is surjective.

Exercise 11:

Let $X = C([0,1])$, $m \in X$ and $T : X \to X$ defined by $Tf = mf$. Determine $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$.

Proof

From Ex. 4 we already know that $\sigma(M_m) = \overline{m([0,1])} = m([0,1])$. Let $\lambda_0 \in m([0,1])$ then there exists $t_0 \in [0,1]$ s.t. $m(t_0) = \lambda_0$. Let $g \in R(\lambda_0 - M_m)$ then there exists $f \in X$ s.t.

$$g(t_0) = (\lambda_0 - M_m)f(t_0) = (\lambda_0 - m(t_0))f(t_0) = 0\,.$$

In other words $\overline{R(\lambda_0 - M_m)} \subseteq \overline{\{g \in X : g(t_0) = 0\}} \neq X$. This means that $\sigma_r(M_m) = R(m)$ and $\sigma_c(M_m) = \emptyset$.
Let $\lambda_0 \in \sigma_p(M_m)$ then there exists $f_0 \in X \setminus \{0\}$ s.t.

$$mf_0 = M_m f_0 = \lambda_0 f_0\,.$$

Function f_0 is continuous and not identically equal to 0, i.e. there must be $0 \leq a \leq b \leq 1$ s.t. $f_0(x) \neq 0$ for all $x \in [a,b]$. This implies $m(x) \stackrel{!}{=} \lambda_0$ for all $x \in [a,b]$. We introduce

$$P(m) := \{\lambda \in \mathbb{C} : 0 \leq \alpha < \beta \leq 1 \text{ s.t.} m(x) = \lambda \,\forall x \in [\alpha,\beta]\}$$

Now we construct eigenfunction $f_0 \in X$ to eigenvalue $\lambda_0 \in P(m)$ as

$$f_0(x) := \begin{cases} 1 & x \in [\alpha,\beta] \\ 0 & x \notin [a,b] \\ [0,1] & otherwise \end{cases}$$

where we have $0 \leq a < \alpha < \beta < b \leq 1$ s.t. $m(x) = \lambda_0$ for all $x \in [a,b]$. We write

$$M_m f_0(x) = m(x) f_0(x) = \lambda_0 f_0(x)$$

which holds for all $x \in [0,1]$. We conclude $\sigma_p(M_m) = P(m)$.

Exercise 12:
Show that the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is well defined automorphism.

Proof
First we prove the following Lemma.

Lemma: Let $(f_n)_n \subseteq \mathcal{S}(\mathbb{R}^d)$ be a sequence s.t. $\lim_{n\to\infty} f_n = f$ in $\mathcal{S}(\mathbb{R}^d)$. Then for all $p \in [1, \infty]$, $\beta \in \mathbb{N}^d$ $\lim_{n\to\infty} f_n = f$ in $L^p(\mathbb{R}^d)$ and

$$\|\partial^\beta g\|_{L^p(\mathbb{R}^d)} \leq C_{p,d} \sum_{\alpha\in\mathbb{N}^d:|\alpha|\leq\left\lceil\frac{d+1}{p}\right\rceil} \rho_{\alpha,\beta}(g)$$

for all $g \in \mathcal{S}(\mathbb{R}^d)$ and given $C_{p,d} > 0$ where $\rho_{\alpha,\beta}(g) := \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta g(x)|$.
Lemma-proof:
First we show that $\forall x \in \mathbb{R}^d$ and $k \in \mathbb{N}_0$:

$$|x|^k \leq C_{d,k} \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=k} |x|^\alpha .$$

We consider a function $h : \mathbb{R}^D \to [0, \infty) : x \mapsto \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=k} |x|\alpha$. Obviously h is continuous. This implies that $h(S^{d-1})$ is compact because a surface of a unit ball in $\mathbb{R}^d$ is compact. The set $h(S^{d-1})$ has maximum and minimum. We denote a minimum by $\kappa_{k,d} := \min h(S^{d-1}) \in (0, \infty)$. The minimum is bigger then zero because $h > 0$ on S^{d-1} which follows from $h(x) = 0 \Leftrightarrow x = 0$. For arbitrary $x \in \mathbb{R}^d$ we have

$$\sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=k} |x|^\alpha = \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=k} \left|\frac{x}{|x|}|x|\right|^\alpha = |x|^k \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=k} \left|\frac{x}{|x|}\right|^\alpha \geq \kappa_{k,d}|x|^k$$

which is equivalent to

$$|x|^k \leq \frac{1}{\kappa_{k,d}} \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=k} |x|^\alpha .$$

The case for $x = 0$ is obvious.
<u>Case $p = \infty$:</u>
Then for $g \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\|\partial^\beta g\|_{L^\infty(\mathbb{R}^d)} = \rho_{0,\beta}(g) \leq \sum_{\alpha\in\mathbb{N}^d:|\alpha|\leq 1} \rho_{\alpha,\beta}(g)$$

i.e. $C_{\infty,d} = 1 > 0$.
<u>Case $p \in [0, \infty)$:</u>
We write

$$\begin{aligned}
\|\partial^\beta g\|_{L^p(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d} |\partial^\beta g(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} = \left(\int_{|x|<1} |\partial^\beta g(x)|^p \mathrm{d}x + \int_{|x|>1} |\partial^\beta g(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq \left(\int_{|x|<1} \|\partial^\beta g(x)\|_\infty^p \mathrm{d}x + \int_{|x|>1} |x|^{d+1}|x|^{-d+1}|\partial^\beta g(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq \left(\mathcal{L}^d(B_1(0))\|\partial^\beta g(x)\|_\infty^p + \sup_{|x|>1} (|x|^{d+1}|\partial^\beta g(x)|^p) \int_{|x|>1} |x|^{-d+1} \mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq \max\left\{\mathcal{L}^d(B_1(0)), \int_{|x|>1} |x|^{-d+1} \mathrm{d}x\right\}^{\frac{1}{p}} \left(\|\partial^\beta g(x)\|_\infty + \sup_{x\in\mathbb{R}^d} (|x|^{\left\lceil\frac{d+1}{p}\right\rceil}|\partial^\beta g(x)|)\right)
\end{aligned}$$

where we used $(a+b)^{\frac{1}{p}} \leq a^{\frac{1}{p}} + b^{\frac{1}{p}}$ for $a, b > 0$ and $p \geq 1$. Now using inequality shown before we get

$$\begin{aligned}
\|\partial^\beta g\|_{L^p(\mathbb{R}^d)} &\leq \max\left\{\mathcal{L}^d(B_1(0)), \int_{|x|>1} |x|^{-d+1}\mathrm{d}x\right\}^{\frac{1}{p}} \left(\|\partial^\beta g(x)\|_\infty + \sup_{x\in\mathbb{R}^d} C_{d,\left\lceil\frac{d+1}{p}\right\rceil} \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=\left\lceil\frac{d+1}{p}\right\rceil} |x|^\alpha |\partial^\beta g(x)|)\right) \\
&\leq \max\left\{\mathcal{L}^d(B_1(0)), \int_{|x|>1} |x|^{-d+1}\mathrm{d}x\right\}^{\frac{1}{p}} \left(\rho_{0,\beta}(g) + C_{d,\left\lceil\frac{d+1}{p}\right\rceil} \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=\left\lceil\frac{d+1}{p}\right\rceil} \sup_{x\in\mathbb{R}^d} |x|^\alpha |\partial^\beta g(x)|)\right) \\
&\leq \max\left\{\mathcal{L}^d(B_1(0)), \int_{|x|>1} |x|^{-d+1}\mathrm{d}x\right\}^{\frac{1}{p}} \left(\rho_{0,\beta}(g) + C_{d,\left\lceil\frac{d+1}{p}\right\rceil} \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=\left\lceil\frac{d+1}{p}\right\rceil} \rho_{\alpha,\beta}(g)\right) \\
&\leq C_{p,d} \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|\leq\left\lceil\frac{d+1}{p}\right\rceil} \rho_{\alpha,\beta}(g)
\end{aligned}$$

□

The Lemma above can be used to show that if a sequence $(f_n) \in \mathcal{S}(\mathbb{R}^d)$ converges in $\mathcal{S}(\mathbb{R}^d)$ it also converges in $L^p(\mathbb{R}^d)$ because

$$\|f_n - f\|_{L^p(\mathbb{R}^d)} \leq C_{p,d} \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|\leq\left\lceil\frac{d+1}{p}\right\rceil} \rho_{\alpha,0}(f_n - f)\,.$$

Now we are ready to show that $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is well defined automorphism.
Well defined: Consider $f \in \mathcal{S}(\mathbb{R}^d)$ then

$$\|(\cdot)^\alpha(\partial^\beta \hat{f})\|_\infty = \frac{(2\pi)^{|\beta|}}{(2\pi)^{|\alpha|}}\|\mathcal{F}[\partial^\alpha(\cdot)^\beta f]\|_\infty \leq \frac{(2\pi)^{|\beta|}}{(2\pi)^{|\alpha|}}\|\partial^\alpha(\cdot)^\beta f\|_1\,.$$

We know that $f \in \mathcal{S}(\mathbb{R}^d)$ implies $x^\alpha\partial^\beta f \in \mathcal{S}(\mathbb{R}^d)$ which gives $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$.
Linear: The integral is linear.
Continuous: Let $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$ then $\forall\alpha, \beta \in \mathbb{N}_0^d$:

$$\begin{aligned}
\rho_{\alpha,\beta}(\hat{f}_n - \hat{f}) &\leq \frac{(2\pi)^{|\beta|}}{(2\pi)^{|\alpha|}}\|\partial^\alpha((\cdot)^\beta(f_n - f))\|_1 \leq C_{1,d}\frac{(2\pi)^{|\beta|}}{(2\pi)^{|\alpha|}} \sum_{\tilde{\alpha}\in\mathbb{N}_0^d:|\tilde{\alpha}|\leq d+2} \rho_{\tilde{\alpha},\alpha}((\cdot)^\beta(f_n - f)) \\
&\leq C_{1,d}\frac{(2\pi)^{|\beta|}}{(2\pi)^{|\alpha|}} \sum_{\tilde{\alpha}\in\mathbb{N}_0^d:|\tilde{\alpha}|\leq d+2} \rho_{\tilde{\alpha}+\beta,\alpha}(f_n - f)
\end{aligned}$$

which converges to 0 as $n \to \infty$ because on the right hand side we have a finite sum.
Bijective: Consider inverse in the form $\mathcal{F}^{-1}(f)(x) = \hat{f}(-x)$. From lecture we have

$$\mathcal{F}^{-1}\mathcal{F}f = \mathcal{F}\mathcal{F}^{-1}f = f$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$ and every $x \in \mathbb{R}^d$. This shows that $\mathcal{F}$ is well defined automorphism on $\mathcal{S}(\mathbb{R}^d)$.