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Spectral Theory

4th Exercise Sheet - Solutions

Exercise 13:
Let $k \in L^1(\mathbb{R}^d)$. Show that the operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined as

$$Tf := k \star f, \quad f \in L^2(\mathbb{R}^d)$$

is well defined, linear and bounded. Determine its spectrum $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$.

Proof
T well defined: We use Young convolution inequality

$$\|f \star g\|_r \leq \|f\|_p \|g\|_q$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$.
T linear: Integral is linear.
T bounded: It is straightforward to see that

$$\|Tf\|_2 \leq \|k\|_1 \|f\|_2$$

i.e. T is continuous. T is continuous and linear which implies bounded.
Spectrum of T: Define $S := \mathcal{F}T\mathcal{F}^{-1} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. Then $S \in \mathcal{L}(L^2(\mathbb{R}^d))$ and

$$Sf = \mathcal{F}T\mathcal{F}^{-1}f = \mathcal{F}(k \star (\mathcal{F}^{-1}f)) = \hat{k}f\,.$$

We have that $\hat{k} \in BUC(\mathbb{R}^d)$, $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$ and S is multiplication operator. Using that $\mathcal{F}$ is isomorphism on $L^2(\mathbb{R}^d)$ we have

$$\sigma(T) = \sigma(S) = \overline{\text{essrange}(\hat{k})} = \overline{R(\hat{k})} = R(\hat{k}) \cup \{0\}\,.$$

Now we need to identify parts of the spectrum.
Point spectrum of T: Let $\lambda \in \mathbb{C}$, $f \in L^2(\mathbb{R}^d)$. Then $0 = (\lambda - T)f$ in $L^2(\mathbb{R}^d)$ is equivalent to

$$0 = \mathcal{F}((\lambda - T)(\mathcal{F}^{-1}f)) = (\lambda - \hat{k})f$$

in $L^2(\mathbb{R}^d)$, i.e. $\hat{k}(x) = \lambda$ or $f(x) = 0$ a.e. in $x \in \mathbb{R}^d$. In other words

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \hat{k} = \lambda \,\text{on}\, U \subseteq R^d \,\text{with}\, \mathcal{L}^d(U) > 0\}$$

with the eigenfunction $f = \chi_V$ where $V \subseteq U$ s.t. $\mathcal{L}^d(V) < \infty$.

Continuous spectrum of T: Now take $\lambda \in \mathbb{C}$, $\lambda \in \overline{R(\hat{k})} = \sigma(T)$ and assume without loss of generality that $\lambda - T$ injective (otherwise λ would be in $\sigma_p(T)$), i.e. $\lambda - T$ can not be surjective. Define a set of zero measure

$$N := \{x \in \mathbb{R}^d : \hat{k} = \lambda\} \subseteq \mathbb{R}^d$$

Consider $x_n \in N$ s.t. $x_n \to x \in \mathbb{R}^d$ then

$$\hat{k}(x) = \hat{k}(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} \hat{k}(x_n) = \lim_{n\to\infty} \lambda = \lambda$$

where we used that $\hat{k}$ is continuous. This means that the set N is closed. We define

$$D := \{\varphi \in \mathcal{S}(\mathbb{R}^d) : \text{supp}(\hat{\varphi}) \subseteq \mathbb{R}^d \setminus N\} .$$

We show later that set D is dense in $L^2(\mathbb{R}^d)$. Choose $g_n \subseteq D$ s.t. $g_n \to g$ in $L^2(\mathbb{R}^d)$ for arbitrary $g \in L^2(\mathbb{R}^d)$ and set

$$f_n := \mathcal{F}^{-1}\left(\chi_{B_n(0)}(x)\frac{\hat{g}_n(x)}{\lambda - \hat{k}(x)}\right) .$$

The function f_n is in $L^2(\mathbb{R}^d)$ because function $\chi_{B_n(0)}(x)\frac{\hat{g}_n(x)}{\lambda-\hat{k}(x)}$ is in $L^2(\mathbb{R}^d)$ due to a compact support of $\chi_{B_n(0)}(x)\hat{g}_n(x)$

$$K := \text{supp}(\chi_{B_n(0)}(x)\hat{g}_n(x)) \subseteq B_n(0) \cap \text{supp}(\hat{g}_n) \subseteq \mathbb{R}^d \setminus N$$

and continuity of $\frac{1}{\lambda-\hat{k}(x)}$ on $\mathbb{R}^d \setminus N$. We estimate

$$\left\|\chi_{B_n(0)}(x)\frac{\hat{g}_n(x)}{\lambda - \hat{k}(x)}\right\|_2 = \left\|\chi_{B_n(0)}(x)\chi_K(x)\frac{\hat{g}_n(x)}{\lambda - \hat{k}(x)}\right\|_2 \leq \left\|\chi_{B_n(0)}(x)\chi_K(x)\frac{1}{\lambda - \hat{k}(x)}\right\|_\infty \|\hat{g}_n(x)\|_2 =$$
$$\leq \left\|\chi_{B_n(0)}(x)\chi_K(x)\frac{1}{\lambda - \hat{k}(x)}\right\|_\infty \|g_n(x)\|_2 < \infty .$$

We take $f_n \in L^2(\mathbb{R}^d)$ then

$$\begin{aligned}\|(\lambda - T)f_n - g\|_2 &= \left\|(\lambda - \hat{k})\hat{f}_n - \hat{g}\right\|_2 = \left\|\chi_{B_n(0)}\hat{g}_n - (\chi_{B_n(0)} + \chi_{B_n(0)^c})\hat{g}\right\|_2 \\ &\leq \|\chi_{B_n(0)}(\hat{g}_n - \hat{g})\|_2 + \|\chi_{B_n(0)^c}\hat{g}\|_2 \leq \|\chi_{B_n(0)}\|_\infty\|\hat{g}_n - \hat{g}\|_2 + \|\chi_{B_n(0)^c}\hat{g}\|_2 \\ &\leq \|g_n - g\|_2 + \|\chi_{B_n(0)^c}\hat{g}\|_2\end{aligned}$$

The last expression converges to 0 because $g_n \to g$ in $L^2(\mathbb{R}^d)$ and $\hat{g} \in L^2(\mathbb{R}^d)$. This means we have a sequence in $R(\lambda - T)$ for arbitrary g, i.e. $R(\lambda - T)$ is dense in $L^2(\mathbb{R}^d)$. We conclude $N = \sigma_c(T)$ and $\sigma_r(T) = \sigma_p(T)$.

Lemma: Let $N \subseteq \mathbb{R}^d$ be closed set of a measure 0. Then

$$D := \{\varphi \in \mathcal{S}(\mathbb{R}^d) : \text{supp}(\hat{\varphi}) \subseteq \mathbb{R}^d \setminus N\} .$$

is dense in $L^2(\mathbb{R}^d)$. *Lemma-proof*: Without loss of generality consider $N \neq \emptyset$ because otherwise $D = \mathcal{S}(\mathbb{R}^d)$ which is dense in $L^2(\mathbb{R}^d)$. The set N^c is open. This implies that for every $n \in \mathbb{N}$ there exists $A_n, B_n \subseteq \mathbb{R}^d$ s.t. $N \subseteq A_n \subseteq \overline{A_n} \subseteq B_n \subseteq \overline{B_n}$ and $0 < d(B_n^c, N) < \frac{1}{n}$. For every $n \in \mathbb{N}$ we choose approximative functions $\eta_n \in C^\infty(\mathbb{R}^d)$ with $\eta_n = 0$ on A_n and $\eta_n = 1$ on B_n^c and $\eta \in [0,1]$ otherwise. For a given $g \in L^2(\mathbb{R}^d)$ we find $(h_k)_{k\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d)$ such that $h_k \to \hat{g}$ in $L^2(\mathbb{R}^d)$ and set $f_k := \mathcal{F}^{-1}h_k$ $\forall k \in \mathbb{N}$. Then

$$\|f_k - g\|_2 = \|\hat{f}_k - \hat{g}\|_2 = \|h_k - \hat{g}\|_2 \to 0$$

for $k \to \infty$. Set also $g_{n,k} := \mathcal{F}^{-1}(\eta_n\hat{f}_k) \in \mathcal{S}(\mathbb{R}^d)$ for every $n, k \in \mathbb{N}$. Furthermore $\hat{f}_k \in C_c^\infty(\mathbb{R}^d)$ and $\eta_n\hat{f}_k \in C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d)$ for every $n, k \in \mathbb{N}$. We have

$$\text{supp}(\hat{g}_{n,k}) = \text{supp}(\eta_n\hat{f}_k) \subseteq A_n^c \subseteq \mathbb{R}^d \setminus N$$

which means $\hat{g}_{n,k} \subseteq D$. We show $\|f_k - g\|_2 < \frac{\epsilon}{2}$ and $\|g_{n,k} - f_k\|_2 < \frac{\epsilon}{2}$ for sufficiently large k and n. The first part follows from use of dominated convergence and

$$\|\mathcal{F}^{-1}(\eta_n\hat{f}) - f\|_2 = \|\eta_n\hat{f} - \hat{f}\|_2 = \|(\eta_n - 1)\hat{f}\|_2 \to 0 .$$

which holds for arbitrary $f \in \mathcal{S}(\mathbb{R}^d)$ with supp$\hat{f}$ compact. This means

$$\|g_{n,k} - g\|_2 \leq \|g_{n,k} - k_k\|_2 + \|f_k - f\|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

for big enough n and k. □

Exercise 14:
Show that for $f, \hat{f} \in L^1(\mathbb{R}^d)$ Fourier inversion formula

$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\xi} \hat{f}(\xi) \mathrm{d}\xi \quad \text{for a. e. } x \in \mathbb{R}^d$$

holds. Furthermore show that $f, \hat{f} \in C_0(\mathbb{R}^d)$ when modified suitably on a null-set.

Proof
We set $h := \exp(-\pi|x|^2)$ and $h_n := n^d h(nx)$ for $\forall x \in \mathbb{R}^d, n \in \mathbb{N}$. Furthermore we also define $f_n := h_n \star f$. Using results about Mollifiers (FA 5.2) we have $f_n \to f$ in $L^1(\mathbb{R}^d)$. We check that $\hat{f}_n = \hat{h}_n \hat{f} \to \hat{f}$ in $L^1(\mathbb{R}^d)$. By a direct calculation one can easily check that

$$\begin{aligned}
\hat{h}_n &= n^d \int_{\mathbb{R}^d} \exp(-2\pi i x\xi - \pi|x|^2)\mathrm{d}x = \prod_{j=1}^{d} n \int_{\mathbb{R}} \exp(-2\pi i x_j\xi_j - \pi|x_j|^2)\mathrm{d}x_j \\
&= \prod_{j=1}^{d} n \exp\left(-\frac{\pi}{n^2}\xi_j^2\right) \int_{\mathbb{R}} \exp\left(-\pi n^2 \left(x_j + i\frac{\xi_j}{n^2}\right)^2\right) \mathrm{d}x_j = \prod_{j=1}^{d} \exp\left(-\frac{\pi}{n^2}\xi_j^2\right) \\
&= \exp\left(-\pi\frac{|\xi|^2}{n^2}\right) .
\end{aligned}$$

We see that $\hat{h}_n \to 1$ pointwise. Now we can write for $\hat{h}_n\hat{f} \in L^1(\mathbb{R}^d)$ the following

$$\exists R > 0 : \int_{B_R(0)^c} |(\hat{h}_n - 1)\hat{f}|\mathrm{d}x \leq 2\int_{B_R(0)^c} |\hat{f}|^2\mathrm{d}x < \frac{\epsilon}{2}$$

and

$$\exists N \forall n > N \forall x \in B_R(0) : |\hat{h}_n(x) - 1| \leq \frac{\epsilon}{2\|\hat{f}\|}$$

which implies $\hat{h}_n\hat{f} \to \hat{f}$ in $L^(\mathbb{R}^d)$. Using Young inequality we obtain

$$\|f_n\|_\infty = \|h_n \star f\|_\infty \leq \|h_n\|_\infty \|f\|_1 < \infty .$$

By (FA 5.2) we have that f_n is continuous. Using inverse Fourier transform on $\hat{f}_n$ we have

$$f_n = \mathcal{F}(\hat{f}_n)(-\cdot) \to \mathcal{F}(\hat{f})(-\cdot)$$

in supremum norm. Due to the continuity of Fourier transform we get $f_n \to f$ in $L^1(\mathbb{R}^d)$ norm. We can conclude that $f = \mathcal{F}(\hat{f})(-\cdot)$ a.e.

Exercise 15: Heisenberg uncertainty principle
Let $d = 1$. Then for arbitrary $\psi \in S(\mathbb{R})$ the inequality

$$\|x\psi\|_2 \|\xi\hat{\psi}\|_2 \geq \frac{1}{4\pi}\|\psi\|_2^2 \,.$$

holds.
Hint: Look at the expression $2\,\mathrm{Re}\langle x\psi, \psi'\rangle$.

Proof
We start by writing

$$\begin{aligned} 2\,\mathrm{Re}\langle x\psi, \psi'\rangle &= \langle x\psi, \psi'\rangle + \langle \psi', x\psi\rangle = \int_{\mathbb{R}} (x\psi\overline{\psi'} + x\overline{\psi}\psi')\mathrm{d}x \\ &= [x|\psi|^2]_{-\infty}^{\infty} - \int_{\mathbb{R}} |\psi|^2 \mathrm{d}x = -\|\psi\|_2^2 \,. \end{aligned}$$

Using Cauchy-Schwarz inequality we obtain

$$\frac{1}{2}\|\psi\|_2^2 = |\,\mathrm{Re}\langle x\psi, \psi'\rangle| \leq \|x\psi\|_2 \|\psi'\|_2 \,.$$

Using the following equality

$$\|\psi'\|_2 = \|\hat{\psi'}\|_2 = 2\pi\|\xi\hat{\psi}\|_2 \,.$$

completes the proof.