Spectral Theory

4th Exercise Sheet - Solutions

Exercise 13:
Let \( k \in L^1(\mathbb{R}^d) \). Show that the operator \( T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) defined as

\[
Tf := k \ast f, \quad f \in L^2(\mathbb{R}^d)
\]

is well defined, linear and bounded. Determine its spectrum \( \sigma_p(T) \), \( \sigma_c(T) \) and \( \sigma_r(T) \).

Proof

\( T \) well defined: We use Young convolution inequality

\[
\| f \ast g \|_r \leq \| f \|_p \| g \|_q
\]

where \( \frac{1}{r} + \frac{1}{q} = \frac{1}{p} + 1 \).

\( T \) linear: Integral is linear.

\( T \) bounded: It is straightforward to see that

\[
\| Tf \|_2 \leq \| k \|_1 \| f \|_2
\]

i.e. \( T \) is continuous. \( T \) is continuous and linear which implies bounded.

Spectrum of \( T \): Define \( S := \mathcal{F}T\mathcal{F}^{-1} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \). Then \( S \in \mathcal{L}(L^2(\mathbb{R}^d)) \) and

\[
Sf = \mathcal{F}T\mathcal{F}^{-1}f = \mathcal{F}(k \ast (\mathcal{F}^{-1}f)) = \hat{k}f.
\]

We have that \( \hat{k} \in BUC(\mathbb{R}^d) \), \( \lim_{|\xi| \to \infty} \hat{f}(\xi) = 0 \) and \( S \) is multiplication operator. Using that \( \mathcal{F} \) is isomorphism on \( L^2(\mathbb{R}^d) \) we have

\[
\sigma(T) = \sigma(S) = \text{essrange}(\hat{k}) = \overline{R(\hat{k})} = R(\hat{k}) \cup \{0\}.
\]

Now we need to identify parts of the spectrum.

Point spectrum of \( T \): Let \( \lambda \in \mathbb{C}, f \in L^2(\mathbb{R}^d) \). Then \( 0 = (\lambda - T)f \) in \( L^2(\mathbb{R}^d) \) is equivalent to

\[
0 = \mathcal{F}((\lambda - T)(\mathcal{F}^{-1}f)) = (\lambda - \hat{k})f
\]

in \( L^2(\mathbb{R}^d) \), i.e. \( \hat{k}(x) = \lambda \) or \( f(x) = 0 \) a.e. in \( x \in \mathbb{R}^d \). In other words

\[
\sigma_p(T) = \{ \lambda \in \mathbb{C} : \hat{k} = \lambda \text{ on } U \subseteq \mathbb{R}^d \text{ with } \mathcal{L}^d(U) > 0 \}
\]

with the eigenfunction \( f = \chi_V \) where \( V \subseteq U \) s.t. \( \mathcal{L}^d(V) < \infty \).

Continuous spectrum of \( T \): Now take \( \lambda \in \mathbb{C}, \lambda \in \overline{R(\hat{k})} = \sigma(T) \) and assume without loss of generality that \( \lambda - T \) injective (otherwise \( \lambda \) would be in \( \sigma_p(T) \)), i.e. \( \lambda - T \) can not be surjective. Define a set of zero measure

\[
N := \{ x \in \mathbb{R}^d : \hat{k} = \lambda \} \subseteq \mathbb{R}^d
\]

Consider \( x_n \in N \) s.t. \( x_n \to x \in \mathbb{R}^d \) then

\[
\hat{k}(x) = \hat{k}(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} \hat{k}(x_n) = \lim_{n \to \infty} \lambda = \lambda
\]
where we used that $\hat{k}$ is continuous. This means that the set $N$ is closed. We define
\[ D := \{ \varphi \in \mathcal{S}(\mathbb{R}^d) : \text{supp}(\varphi) \subseteq \mathbb{R}^d \setminus N \} . \]
We show later that set $D$ is dense in $L^2(\mathbb{R}^d)$. Choose $g_n \subseteq D$ s.t. $g_n \to g$ in $L^2(\mathbb{R}^d)$ for arbitrary $g \in L^2(\mathbb{R}^d)$ and set
\[ f_n := F^{-1} \left( \chi_{B_n(0)}(x) \frac{\hat{g}_n(x)}{\lambda - \hat{k}(x)} \right) . \]
The function $f_n$ is in $L^2(\mathbb{R}^d)$ because function $\chi_{B_n(0)}(x) \frac{\hat{g}_n(x)}{\lambda - \hat{k}(x)}$ is in $L^2(\mathbb{R}^d)$ due to a compact support of $\chi_{B_n(0)}(x)\hat{g}_n(x)$
and continuity of $\frac{1}{\lambda - k(x)}$ on $\mathbb{R}^d \setminus N$. We estimate
\[ \left\| \chi_{B_n(0)}(x) \frac{\hat{g}_n(x)}{\lambda - \hat{k}(x)} \right\|_2 = \left\| \chi_{B_n(0)}(x) \chi_K(x) \frac{\hat{g}_n(x)}{\lambda - \hat{k}(x)} \right\|_2 \leq \left\| \chi_{B_n(0)}(x) \chi_K(x) \frac{1}{\lambda - \hat{k}(x)} \right\|_\infty \| \hat{g}_n(x) \|_2 = \left\| \chi_{B_n(0)}(x) \chi_K(x) \frac{1}{\lambda - \hat{k}(x)} \right\|_\infty \| g_n(x) \|_2 < \infty . \]
We take $f_n \in L^2(\mathbb{R}^d)$ then
\[ \left\| (\lambda - T)f_n - g \right\|_2 = \left\| (\lambda - \hat{k})\hat{f}_n - \hat{g} \right\|_2 = \left\| \chi_{B_n(0)}(0)\hat{g}_n - (\chi_{B_n(0)}(x) + \chi_{B_n(0)^c})\hat{g} \right\|_2 \leq \| \chi_{B_n(0)}(0)\hat{g}_n - \hat{g} \|_2 + \| \chi_{B_n(0)}\hat{g} \|_2 \leq \| \hat{g}_n - \hat{g} \|_2 + \| \chi_{B_n(0)}\hat{g} \|_2 \]
The last expression converges to 0 because $g_n \to g$ in $L^2(\mathbb{R}^d)$ and $\hat{g} \in L^2(\mathbb{R}^d)$. This means we have a sequence in $R(\lambda - T)$ for arbitrary $g$, i.e. $R(\lambda - T)$ is dense in $L^2(\mathbb{R}^d)$. We conclude $N = \sigma_c(T)$ and $\sigma_r(T) = \sigma_p(T)$. 

**Lemma:** Let $N \subseteq \mathbb{R}^d$ be closed set of a measure 0. Then
\[ D := \{ \varphi \in \mathcal{S}(\mathbb{R}^d) : \text{supp}(\varphi) \subseteq \mathbb{R}^d \setminus N \} . \]
is dense in $L^2(\mathbb{R}^d)$. **Lemma-proof:** Without loss of generality consider $N \neq \emptyset$ because otherwise $D = \mathcal{S}(\mathbb{R}^d)$ which is dense in $L^2(\mathbb{R}^d)$. The set $N^c$ is open. This implies that for every $n \in N$ there exists $A_n, B_n \subseteq \mathbb{R}^d$ s.t. $N \subseteq A_n \subseteq B_n \subseteq \mathbb{R}^d$ and $0 < d(B_n, N) < \frac{1}{n}$. For every $n \in N$ we choose approximative functions $\eta_n \in C(\mathbb{R}^d)$ with $\eta_n = 0$ on $A_n$ and $\eta_n = 1$ on $B_n^c$ and $\eta \in [0, 1]$ otherwise. For a given $g \in L^2(\mathbb{R}^d)$ we find $(\eta_k)_{k \in \mathbb{N}} \subseteq C(\mathbb{R}^d)$ such that $h_k \to g$ in $L^2(\mathbb{R}^d)$ and set $f_k := F^{-1}h_k \forall k \in \mathbb{N}$. Then
\[ \left\| f_k - g \right\|_2 = \left\| \hat{f}_k - \hat{g} \right\|_2 = \left\| \hat{g}_n - \hat{g} \right\|_2 = \left\| \hat{g}_n - \hat{g} \right\|_2 \to 0 \]
for $k \to \infty$. Set also $g_{n,k} := F^{-1}(\eta_n\hat{f}_k) \in \mathcal{S}(\mathbb{R}^d)$ for every $n, k \in \mathbb{N}$. Furthermore $\hat{f}_k \in C(\mathbb{R}^d)$ and $\eta_n\hat{f}_k \in C(\mathbb{R}^d)$ for every $n, k \in \mathbb{N}$. We have
\[ \text{supp}(\hat{g}_{n,k}) = \text{supp}(\eta_n\hat{f}_k) \subseteq A_n^c \subseteq \mathbb{R}^d \setminus N \]
which means $\hat{g}_{n,k} \subseteq D$. We show $\left\| f_k - g \right\|_2 < \frac{\epsilon}{2}$ and $\left\| g_{n,k} - f \right\|_2 < \frac{\epsilon}{2}$ for sufficiently large $k$ and $n$. The first part follows from use of dominated convergence and
\[ \left\| F^{-1}(\eta_n\hat{f}) - f \right\|_2 = \| \eta_n\hat{f} - \hat{f} \|_2 = \| (\eta_n - 1)\hat{f} \|_2 \to 0 . \]
which holds for arbitrary $f \in \mathcal{S}(\mathbb{R}^d)$ with supp$\hat{f}$ compact. This means
\[ \left\| g_{n,k} - g \right\|_2 \leq \left\| g_{n,k} - k_k \right\|_2 + \left\| f_k - f \right\|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \]
for big enough $n$ and $k$. □
Exercise 14:
Show that for $f, \hat{f} \in L^1(\mathbb{R}^d)$ Fourier inversion formula
\[ f(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \hat{f}(\xi)} \, d\xi \quad \text{for a.e. } x \in \mathbb{R}^d \]
holds. Furthermore show that $f, \hat{f} \in C_0(\mathbb{R}^d)$ when modified suitably on a null-set.

Proof
We set $h := \exp(-\pi|x|^2)$ and $h_n := n^d h(nx)$ for $\forall x \in \mathbb{R}^d, n \in \mathbb{N}$. Furthermore we also define $f_n := h_n \ast f$. Using results about Mollifiers (FA 5.2) we have $f_n \to f$ in $L^1(\mathbb{R}^d)$. We check that $\hat{f}_n := \hat{h}_n \ast \hat{f} \to \hat{f}$ in $L^1(\mathbb{R}^d)$. By a direct calculation one can easily check that
\[
\hat{h}_n = n^d \int_{\mathbb{R}^d} \exp(-2\pi i x \cdot \hat{f}) \, dx = \prod_{j=1}^d n \int_{\mathbb{R}} \exp(-2\pi i x_j \xi_j - \pi |x_j|^2) \, dx_j = \prod_{j=1}^d \int_{\mathbb{R}} \exp \left( -\frac{\pi |\xi_j|^2}{n^2} \right) \, dx_j = \exp \left( -\frac{\pi |\xi|^2}{n^2} \right).
\]
We see that $\hat{h}_n \to 1$ pointwise. Now we can write for $\hat{h}_n \hat{f} \in L^1(\mathbb{R}^d)$ the following
\[
\exists R > 0 : \int_{B_R(0)^c} |(\hat{h}_n - 1) \hat{f}| \, dx \leq 2 \int_{B_R(0)^c} |\hat{f}|^2 \, dx < \frac{\epsilon}{2}
\]
and
\[
\exists N \forall n > N \forall x \in B_R(0) : |\hat{h}_n(x) - 1| \leq \frac{\epsilon}{2\|\hat{f}\|}
\]
which implies $\hat{h}_n \hat{f} \to \hat{f}$ in $L(\mathbb{R}^d)$. Using Young inequality we obtain
\[
\|f_n\|_\infty = \|h_n \ast f\|_\infty \leq \|h_n\|_\infty \|f\|_1 < \infty.
\]
By (FA 5.2) we have that $f_n$ is continuous. Using inverse Fourier transform on $\hat{f}_n$ we have
\[
f_n = \mathcal{F}(\hat{f}_n)(-\cdot) \to \mathcal{F}(\hat{f})(-\cdot)
\]
in supremum norm. Due to the continuity of Fourier transform we get $f_n \to f$ in $L^1(\mathbb{R}^d)$ norm. We can conclude that $f = \mathcal{F}(\hat{f})(-\cdot)$ a.e.
Exercise 15: Heisenberg uncertainty principle
Let $d = 1$. Then for arbitrary $\psi \in S(\mathbb{R})$ the inequality
\[
\|x\psi\|_2 \|\xi \hat{\psi}\|_2 \geq \frac{1}{4\pi} \|\psi\|^2.
\]
holds.

*Hint:* Look at the expression $2 \text{Re}(x\psi, \psi')$.

**Proof**

We start by writing
\[
2 \text{Re}(x\psi, \psi') = \langle x\psi, \psi' \rangle + \langle \psi', x\psi \rangle = \int_{\mathbb{R}} (x\overline{\psi'} + x\overline{\psi}\psi')dx
\]
\[
= [x|\psi|^2]_{-\infty}^{\infty} - \int_{\mathbb{R}} |\psi|^2dx = -\|\psi\|^2.
\]

Using Cauchy-Schwarz inequality we obtain
\[
\frac{1}{2} \|\psi\|^2 = |\text{Re}(x\psi, \psi')| \leq \|x\psi\|_2 \|\psi'\|_2.
\]

Using the following equality
\[
\|\psi'\|_2 = \|\hat{\psi'}\|_2 = 2\pi \|\xi\hat{\psi}\|_2.
\]
completes the proof.