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Spectral Theory

5th Exercise Sheet - Solutions

Exercise 16: Show the following:
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $T \in \mathscr{L}(X, Y)$ be a Fredholm operator. Then $T' \in \Phi(Y', X')$ and $\mathrm{ind}(T) = -\mathrm{ind}(T')$.

Proof
Let $T \in \mathscr{L}(X, Y)$. Then

$$\alpha(T') = \dim(N(T')) = \dim((Y/_{R(T)})') = \dim(Y/_{R(T)}) = \beta(T) < \infty$$
$$\beta(T') = \dim(X'/_{R(T')}) = \dim((N(T))') = \dim(N(T)) = \alpha(T) < \infty$$

where we used Closed range theorem and the fact that for finite dimensional vector spaces the dual space have the same dimension as the original space. Using Closed range theorem we know that $R(T') \subseteq X'$ is closed. This gives $T' \in \mathscr{L}(Y', X')$. Furthermore

$$\mathrm{ind}(T') = \alpha(T') - \beta(T') = \beta(T) - \alpha(T) = -\mathrm{ind}(T)\,.$$

Exercise 17: Show the following:
Let A be a closed operator in a Banach space X, $\lambda_0 \in \rho(A)$ and $z \in \mathbb{C} \setminus \{\lambda_0\}$. Then for all $k \in \mathbb{N}$ we have

$$N\left((z-A)^k\right) = N\left(\left((\lambda_0 - z)^{-1} - R(\lambda_0, A)\right)^k\right), \; R\left((z-A)^k\right) = R\left(\left((\lambda_0 - z)^{-1} - R(\lambda_0, A)\right)^k\right).$$

In particular, $z - A \in \Phi([D(A)], X)$ if and only if $(\lambda_0 - z)^{-1} - R(\lambda_0, A) \in \Phi(X)$. In this case one has $\operatorname{ind}(z - A) = \operatorname{ind}((\lambda_0 - z)^{-1} - R(\lambda_0, A))$.

Proof
We write

$$z - A = z - \lambda_0 + \lambda_0 - A = ((z-\lambda_0)R(\lambda_0, A) + I)(\lambda_0 - A) = \left(R(\lambda_0, A) + \frac{1}{z - \lambda_0}\right)(z-\lambda_0)(\lambda_0 - A)$$

where $z - \lambda_0$ and $\lambda_0 - A$ are bijection from $D(A)$ to X. Then $N(z-A) = N\left(R(\lambda_0, A) + \frac{1}{z-\lambda_0}\right)$ and $R(z-A) = R\left(R(\lambda_0, A) + \frac{1}{z-\lambda_0}\right)$. Similarly we can write

$$(z-A)^k = [(z-\lambda_0)R(\lambda_0, A) + I)(\lambda_0 - A)]^k = \left(R(\lambda_0, A) + \frac{1}{z - \lambda_0}\right)^k (z-\lambda_0)^k (\lambda_0 - A)^k$$

and use the same arguments as before.

Exercise 18:

1. Let R, L be a right and left shift operator respectively defined on $l^2(\mathbb{N})$. For which $\lambda \in \mathbb{C}$ one has $\lambda - R \in \Phi(l^2)$ and $\lambda - L \in \Phi(l^2)$. What are $\mathrm{ind}(\lambda - L)$ and $\mathrm{ind}(\lambda - R)$ for such λ respectively?
2. Let $A = \frac{d^2}{dx^2}$ and consider $A : C^2[0,1] \to C[0,1]$. For which $\lambda \in \mathbb{C}$ the operator $\lambda - A$ is Fredholm? Determine $\mathrm{ind}(\lambda - A)$ for these λ.

Proof

1. We note that the adjoint operator to L is R. This means that it is enough to investigate only one of them. We know that $\sigma(L) = \{|\lambda| \le 1\}$ and $\sigma\{|\lambda| < 1\}$. Furthermore $\|L\| = 1$ and $r(L) = 1$. Consider $\phi \in C^2([0,1])$. Then $\phi \in N(\lambda - A)$ is equivalent to

$$\lambda x - Lx = 0 \Leftrightarrow \lambda x_n = x_{n+1} \Leftrightarrow x_{n+1} = \lambda^n x_1 \,.$$

This implies $\sigma_p(L) = \{|\lambda| < 1\}$ and $\sigma_{ess}(L) = \{|\lambda| = 1\}$ because $a_n = \lambda^{n-1} \in l^2$ if $|\lambda| < 1$ and not in l^2 for $|\lambda| = 1$.
<u>$N(\lambda - L)$</u>: From above we see that $\alpha(\lambda - L) = 1$ for $|\lambda| < 1$. From the fact that $\|L\| = 1$ we get $\alpha(\lambda - L) = 0$ for $|\lambda| > 1$ due to bijectivity of $\lambda - L$.
<u>$R(\lambda - A)$</u>: We know $\rho(L) = \{|\lambda| > 1\}$. This implies that $\lambda - L$ is bijection on l^2 if $|\lambda| > 1$, i.e. $\beta(\lambda - L) = 0$ for $|\lambda| > 1$. We remark that $L : X_0 := \{a_n \in l^2 : a_1 = 0\} \to l^2$ is a bijection and $(L|_{X_0})^{-1} = R : l^2 \to X_0$. Now consider for $\lambda \in \mathbb{C}$ s.t. $|\lambda| < 1$ the operator $T := \sum_{k=0} \infty \lambda^k R^{k+1}$. Operator T converge absolutely in operator norm. Furthermore one can easily check

$$(\lambda - L)T = \sum_{k=0} \infty \lambda^{k+1} R^{k+1} - \sum_{k=0} \infty \lambda^k R^k = -I_{l^2}$$

which implies $R(\lambda - L) = l^2$. Together with the previous result we have $\beta(\lambda - L) = 0$ for $|\lambda| \ne 1$. We conclude

$$|\lambda| < 1 \,, \quad \mathrm{ind}(\lambda - L) = 1 \,, \qquad\qquad |\lambda| > 1 \,, \quad \mathrm{ind}(\lambda - L) = 0 \,.$$

Due to the continuity of the index for perturbation of Fredholm operators by bounded operators $\lambda - L$ can not be Fredholm for $|\lambda| = 1$.
2. <u>$N(\lambda - A)$</u>:

$$\lambda \phi = \phi'' \,.$$

This equation has solution in the form

$$\lambda = 0 \,, \quad \psi = 1 \text{ and } \psi = x \,, \qquad\qquad \lambda \ne 0 \,, \quad \psi = \exp(\pm\sqrt{\lambda} x)$$

This gives $\alpha(\lambda - A) = 2$. <u>$R(\lambda - A)$</u>: We need to solve

$$\lambda \phi - \phi'' = g$$

for given $g \in C^0([0,1])$. From the theory of ODE we know that ther always exists a solution, i.e. $R(\lambda - A) = C([0,1])$. This means $\beta(\lambda - A) = 0$. We conclude that for each $\lambda \in \mathbb{C}$ $\mathrm{ind}(\lambda - A) = 2$.