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Spectral Theory

6th Exercise Sheet - Solutions

Exercise 19:
Let X, Y, Z be Banach spaces, $T \in \Phi(X,Y)$ and $S \in \Phi(Y,Z)$. Show that $ST \in \Phi(X,Z)$ and $\text{ind}(ST) = \text{ind}(S) + \text{ind}(T)$.
Hint: First find a complement N of $N(T)$ in $N(ST)$ and a complement N_S of $T(N)$ in $N(S)$. Then represent X and Y as direct sums.

Proof
We start by finding the orthogonal complement as suggested in hint

$$N(ST) = N(T) \oplus N\,.$$

This means

$$T(N) = R(T) \cap N(S) = \{Tx | x \in X, STx = 0\}\,.$$

Next we find a complement of $T(N)$ in $N(S)$ as

$$N(S) = T(N) \oplus N_S\,.$$

Now we introduce G_0 as a complement in X

$$X = N(T) \oplus N \oplus G_0\,.$$

The action of T on X yields

$$\{0\} \oplus T(N) \oplus T(G_0) \oplus N_S \oplus H_1 = Y$$

where H_1 is a complement of $R(T) \oplus N_S$ in Y. Similarly by action of S on Y

$$\{0\} \oplus R(ST) \oplus \{0\} \oplus S(H_1) \oplus H_S = Z$$

where H_S is a complement of $R(ST) \oplus S(H_1) = R(S)$ in Z. Due to the construct we can check that $\alpha(ST) < \infty$ and $\beta(ST) < \infty$. From above we have

$$\begin{aligned}
&\beta(ST) = \dim S(H_1) + \beta(S)\,,\\
&\dim S(H_1) = \dim H_1 = \text{codim} R(T) - \dim N_S = \beta(T) - \dim N_S\,,\\
&\alpha(S) = \dim T(N) + \dim N_S = \dim N + \dim N_S = \alpha(ST) - \alpha(T) + \dim N_S\,,\\
&\alpha(ST) = \alpha(S) + \alpha(T) - \dim N_S\,,\\
&\beta(ST) = \beta(T) - \dim N_S + \beta(S)\,.
\end{aligned}$$

This implies $\text{ind}(ST) = \text{ind}(S) + \text{ind}(T)$.

Exercise 20: Show the following:

Let A be a linear operator in $\mathbb{C}^n$ given by $A = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}$ where $\lambda \in \mathbb{C}$. Find a representation for the operator $f(A)$ (given by the Dunford functional calculus), where f is holomorphic on a neighborhood of λ.

Proof

Clearly A is a bounded operator. We denote $U \subseteq \mathbb{C}$ an open neighborhood of $\sigma(A) = \{\lambda\}$. We find a curve Γ s.t. $n(z,\Gamma) = 1$ for $z \in U$ and $n(z,\Gamma) = 0$ for $z \notin U$. Furthermore, we assume that f is holomorphic in U. Then

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(\mu)(\mu - A)^{-1}\mathrm{d}\mu\,.$$

We rewrite the resolvent of A as

$$(\mu - A)^{-1} = (\mu - \lambda - N)^{-1} = (\mu - \lambda)^{-1}\left(I - \frac{N}{\mu - \lambda}\right)^{-1} = (\mu - \lambda)^{-1}\sum_{k=0}^{n-1}\left(\frac{N}{\mu - \lambda}\right)^k$$

where $N = A - \lambda I$. We rewrite

$$f(A) = \sum_{k=0}^{n-1}\frac{1}{2\pi i}\int_\Gamma f(\mu)(\mu - \lambda)^{-k-1}\mathrm{d}\mu N^k = \sum_{k=0}^{n-1}\frac{f^{(k)}(\lambda)}{k!}\mathrm{d}\mu N^k$$

i.e.

$$f(A) = \begin{pmatrix} f(\lambda) & \frac{f'(\lambda)}{1!} & \frac{f''(\lambda)}{2!} & \cdots \\ & \ddots & \ddots & \\ & & f(\lambda) & \frac{f'(\lambda)}{1!} \\ & & & f(\lambda) \end{pmatrix}$$

Exercise 21:
Let $X = L^2(\mathbb{R}^+)$ and let $T : X \to X$ be defined by

$$Tf(x) := \frac{1}{x}\int_0^x f(y)dy \quad \text{for} \quad f \in X, x \in \mathbb{R}^+$$

Show that:

1. T is well defined, linear and continuous.
2. T is not compact.
   Hint: You can use Fredholm alternative.

Proof