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Spectral Theory

7th Exercise Sheet - Solutions

Exercise 22:
Show the following:
Let A be a closed linear operator on a Banach Space X and

$$\mathcal{D}(A^{k+1}) := \{x \in \mathcal{D}(A^k) : Ax \in \mathcal{D}(A^k)\}\,.$$

Then $\mathcal{D}(A^k)$ is a Banach space for the norm

$$|||x||| := \sum_{j=0}^{k} \|A^j x\|\,.$$

Proof
We prove this in an inductive way. Let x_n be $|||\cdot|||$-Cauchy in $\mathcal{D}(A^k)$ then $\|A^j x_n\|$ is $\|\cdot\|$-Cauchy in X. Furthermore

$$\begin{aligned} x_n &\to y_0 \\ Ax_n &\to y_1 \end{aligned}$$

then by closedness of A we obtain $y_0 \in \mathcal{D}(A)$ and $Ay_0 = y_1$. Next

$$\begin{aligned} Ax_n &\to y_1 \\ A^2 x_n &\to y_2 \end{aligned}$$

along with closedness of A implies $Ay_0 \in \mathcal{D}(A)$ and $A^2 y_0 = y_2$. Using the definition of y_0 yields $y_0 \in \mathcal{D}(A^2)$. This can be repeated up to $y_0 \in \mathcal{D}(A^k)$.

Exercise 23:
Show the following:

1. Let A be a closed linear operator on a Banach Space X s.t. $\rho(A) \neq \emptyset$. Then A^k is closed for any $k \in \mathbb{N}$.
2. In particular, the graph norm of A^k is equivalent to the norm of Ex. 22.

Proof
We have $\rho(A) \neq \emptyset$, i.e. there exists $\lambda_0 \in \rho(A)$ s.t. $R(\lambda_0, A) : X \to [\mathcal{D}(A)]$ is an isomorphism. Using this we can show that $\|x\| + \|Ax\| \sim \|(\lambda_0 - A)x\|$. One direction is a direct consequence of triangular inequality. The opposite direction follows from injectivity and surjectivity of $R(\lambda_0, A)$. We note

$$X \underset{R(\lambda_0,A)}{\overset{\lambda_0 - A}{\leftrightarrows}} [D(A)] \, .$$

We distinguish two cases.
$0 \in \rho(A)$: Using the above we have $\|x\| + \|Ax\| \sim \|Ax\|$. We note that

$$[D(A)] \overset{R(\lambda_0,A)}{\longrightarrow} (D(A^2), |||\cdot|||) \text{ and } R(\lambda_0, A)^2 : X \to (D(A^2), |||\cdot|||) \, .$$

This implies $\|(\lambda_0 - A)^2 x\| \sim \|x\| + \|Ax\| + \|A^2x\|$. Using the fact that $\lambda_0 = 0$ we obtain $\|A^2 x\| \sim \|x\| + \|Ax\| + \|A^2 x\|$.
$0 \notin \rho(A)$: Then we introduce a shifted operator $B := \lambda_0 - A$ s.t. $0 \in \rho(B)$ and use

$$A^2 = A^2 I_X = A^2(\lambda_0 - A)^2 R(\lambda_0, A)^2 = (\lambda_0 - A)^2 A^2 R(\lambda_0, A)^2$$

where on the right hand side we have product of closed operator B and bounded operator.

Exercise 24:
Let A be a closed linear operator on a Banach Space X. Then

$$\sigma(A^k) = \{\lambda^k : \lambda \in \sigma(A)\}\,.$$

Proof
Fix λ and write $p(u) := \mu^k - z = \prod_{j=1}^k (\mu - \mu_j)$. Then $A^k - z = \prod_{j=1}^k (A - \mu_j)$. We take $z = \lambda^k$ then there exists μ_j s.t. $\mu_j^k = \lambda^k$. Then
$\subseteq$: Let $z \in \sigma(A^k)$ then $A^k - z$ is not injective/surjective. This implies that one of the operators $A - \mu_j$ is not injective/surjective. This can be shown in the following way
$\rho(A) = \emptyset$: This implies that every $A - \mu_j$ is not injective $\sigma(A) = \mathbb{C}$.
$\rho(A) \neq \emptyset$: We can write

$$\begin{array}{ccc} D(A) & \overset{A-\mu_j}{\longrightarrow} & X \\ (\lambda_0 - A)^k \uparrow & & \downarrow R(\lambda_0, A)^k \\ D(A^{k+1}) & \underset{A-\mu_j}{\longrightarrow} & D(A^k) \end{array}$$

$\supseteq$: Let $\mu_j \in \sigma(A)$ then $A - \mu_j$ is not injective and $\lambda^k \in \sigma(A^k)$. We can use the same argument for arbitrary power of A.

Exercise 25:
Let $1 \leq k < \infty$ and $X \times Y$ be σ-finite product measure space with measures dx and dy respectively. Then for a measurable function $F(x,y)$

$$\left(\int_Y \left(\int_X |F(x,y)| \mathrm{d}x \right)^k \mathrm{d}y \right)^{\frac{1}{k}} \leq \int_X \left(\int_Y |F(x,y)|^k \mathrm{d}y \right)^{\frac{1}{k}} \mathrm{d}x$$

holds. Furthermore equality holds for $k = 1$ or $F(x,y) = \phi(x)\psi(y)$ if $k > 1$.

Proof
We prove this claim using following lemma.

Lemma: If $k > 1$ then a necessary and sufficient condition that

$$\int |f|^k \mathrm{d}x \leq F$$

is that

$$\int fg \, \mathrm{d}x \leq F \leq F^{\frac{1}{k}} G^{\frac{1}{k'}}$$

for all g s.t. $\int g^{k'} \leq G$ where $\frac{1}{k} + \frac{1}{k'} = 1$.

case $k = 1$: The claim follows from Fubini Theorem.
case $k > 1$: Define $J := J(x) = \int f(x,y) dy$. Then using Lemma we get

$$\int J^k \mathrm{d}x \leq M^k \Leftrightarrow \int Jg \leq M^k$$

for all g s.t. $\int g^{k'} \leq 1$. Using Cauchy-Schwarz inequality we write

$$\int Jg \mathrm{d}x = \int g(x) \int f(x,y) \mathrm{d}y \mathrm{d}x = \int \left(\int g(x) f(x,y) \mathrm{d}x \right) \mathrm{d}y \leq \int \left(\int f^k \mathrm{d}x \right)^{\frac{1}{k}} \mathrm{d}y \, .$$

We can conclude that $M = \int \left(\int f^k \mathrm{d}x \right)^{\frac{1}{k}} \mathrm{d}y$.

Proof-Lemma: We prove this using contraposition. If $\int f^k \mathrm{d}x > F$ then there exists sufficiently large n s.t. $\int (f)_n^k \mathrm{d}x > F$. We choose g to be effectively proportional to $(f_n)^{k-1}$ then

$$\int fg \mathrm{d}x \geq \int (f)_n g \mathrm{d}x = \left(\int (f)_n^k \mathrm{d}x \right)^{\frac{1}{k}} G^{\frac{1}{k'}} > F^{\frac{1}{k}} G^{\frac{1}{k'}} \, .$$

which is contradiction with the hypothesis of the theorem.