Exercise 26:
Show the following:
Let $B \in S_2(H)$. Then $B^* \in S_2(H)$ and furthermore

$$\|S\|_2^2 = \sum_{j=1}^{\infty} \|Be_j\|^2 < \infty$$

for any orthonormal basis $E = (e_n) \subseteq H$ with the sum independent of $E$.

**Proof**
We denote $N_E(B) := \sum_{j=1}^{\infty} \|Be_j\|^2$. Using Parseval relation we can write

$$\|Be_j\|^2 = \sum_{k=1}^{\infty} |(f_k|Be_j)|^2 = \sum_{k=1}^{\infty} |(B^* f_k|e_j)|^2$$

where the sums converge absolutely. Using the latter we can rewrite $N_E(B)$ as

$$N_E(B) = \sum_{j,k=1}^{\infty} |(e_j|B^* f_k)|^2 = \sum_{k=1}^{\infty} \|B^* f_k\|^2 = N_{E_B}(B^*)$$

where $E_B$ is a basis for which we know $\sum_{k=1}^{\infty} \|B f_k\|^2 < \infty$. To show the independence on a basis we start with $E = E_B$ which implies $N_{E_B}(B) = N_{E_B}(B^*) < \infty$, i.e. $B^* \in S_2$. Using the identity above we get $N_E(B) = N_{E_B}(B)$ for arbitrary basis $E$. 
Exercise 27:
Show the following:
1. $S_2(H)$ is a two–sided $\star$–ideal in $\mathcal{L}(H)$ and $\| \cdot \|_{\nu_2}$ is a norm on it which fulfills 
   $$\| S \|_{\nu_2} \geq \| S \|_2$$
   for all $S \in S_2(H)$. Recall that Hilbert–Schmidt operator is compact, i.e. $S_2(H) \subseteq \mathcal{K}(H)$.
2. $S_2(H)$ equipped with the inner product 
   $$(B|C)_2 = \sum_{j=1}^{\infty} (B e_j | C e_j)_H$$
   is a Hilbert space.

Proof

1. Norm: First we start by showing $\| \cdot \|_{\nu_2}$ is a norm. We note 
   $$\| (B + C) e_j \|_2^2 \leq \| B e_j \|_2^2 + 2 \| B e_j \| \| C e_j \| + \| C e_j \|_2^2$$
   and 
   $$(a + b)^s \leq a^s + b^s$$
   for all $a, b \geq 0$ with $0 \leq s \leq 1$. This implies 
   $$\| B + C \|_{\nu_2} \leq \left( \sum_{j=1}^{\infty} (\| B e_j \| + \| C e_j \|)^2 \right)^{\frac{1}{2}} \leq \| B \|_{\nu_2} + \| C \|_{\nu_2}.$$ 
   It is obvious that 
   $$\| \alpha B \|_{\nu_2} = |\alpha| \| B \|_{\nu_2}.$$ 
   Furthermore 
   $$\| B \|_{\nu_2} = 0 \Rightarrow \forall j : B e_j = 0$$
   which implies that $B = 0$ by continuity. This proves that $(S_2, \| \cdot \|_{\nu_2})$ is a normed space.

Ideal: Let $D \in \mathcal{L}(H)$ and $B \in S_2(H)$ then we have 
   $$\| DB e_j \| \leq \| D \| \| B e_j \|$$
   which gives $DB \in S_2$. From previous exercise we know that $C \in S_2(H)$ implies $C^* \in S_2(H)$. We have $D^* B^* \in S_2$ because adjoint of a bounded operator is bounded. It can be rewritten as $BD = (D^* B^*)^* \in S_2$.

Compactness: Consider a sequence of projectors $P_n$ onto $\text{lin}\{e_1, \ldots, e_n\}$ and define finite rank operators $B_n := P_n B$. Then

$$\| (B - B_n) x \|_2^2 = \sum_{j=n+1}^{\infty} |(e_j | B x)|^2 \leq \| x \|_2^2 \sum_{j=n+1}^{\infty} \| B^* e_j \|_2^2.$$ 

Using $B^* \in S_2$ we obtain $B_n \to B$. The limit implies that $B$ is compact because it is a limit of finite rank operators.
2. **Scalar product**: The direct calculation shows that $\| \cdot \|_{\nu_2}$ satisfies parallelogram identity. Furthermore it is easy to check that $\| \cdot \|_{\nu_2}$ is generated by $(B, C)_{\nu_2}$.

**Completeness**: Consider a $\nu_2$-Cauchy sequence of Hilbert Schmidt operators $B_n$. Using the first part and the fact that $K(H)$ is a closed subspace of $\mathcal{L}(H)$ we obtain that $B_n$ is also a Cauchy sequence in operator norm which converges to certain compact operator. We need to check that the limiting operator $B$ is Hilbert Schmidt. For a given $\epsilon > 0$ and an orthonormal basis we have

$$\sum_{j=n+1}^{N} \| (B - B_n)e_j \|^2 \leq \| B - B_n \|^2_{\nu_2} < \epsilon$$

for large enough $m, n$ and any $N = 1, 2, \ldots$. Taking first the limit $n \to \infty$ and then $N \to \infty$ we get $B \in S_2$ and $\| B - B_m \| < \epsilon$ for sufficiently large $m$, i.e. $\lim_{m \to \infty} \| B - B_m \| = 0$. 

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Exercise 28:
Show the following:
Any trace-class operator $S$ is Hilbert–Schmidt, and therefore compact, $S_1(H) \subseteq S_2(H) \subseteq \mathcal{K}(H)$ and
\[
\text{Tr}(|S|) \geq \|S\|_{\nu_2} \geq \|S\|_2.
\]

Proof
Consider $A \in S_1$ then
\[
(\text{Tr}|A|)^2 \geq \|A\| \text{Tr}|A| \geq \text{Tr}|A|^2 = \text{Tr}A^*A = \|A\|_{\nu_2}^2.
\]
This implies $A \in S_1$ then also $|A|^2 \in S_1$ and $A \in S_2$. 

http://www.math.kit.edu/iana1/edu/spectraltheo2019s/en