Institut für Analysis
PD Dr. Peer Christian Kunstmann
Dr. Michal Jex

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Spectral Theory

8th Exercise Sheet - Solutions

Exercise 26:
Show the following:
Let $B \in S_2(H)$. Then $B^* \in S_2(H)$ and furthermore

$$\|S\|_{\nu_2}^2 = \sum_{j=1}^{\infty} \|Be_j\|^2 < \infty$$

for any orthonormal basis $E = (e_n) \subseteq H$ with the sum independent of E.

Proof
We denote $N_{\mathcal{E}}(B) := \sum_{j=1}^{\infty} \|Be_j\|^2$. Using Parseval relation we can write

$$\|Be_j\|^2 = \sum_{k=1}^{\infty} |(f_k|Be_j)|^2 = \sum_{k=1}^{\infty} |(B^* f_k|e_j)|^2$$

where the sums converge absolutely. Using the latter we can rewrite $N_{\mathcal{E}}(B)$ as

$$N_{\mathcal{E}}(B) = \sum_{j,k=1}^{\infty} |(e_j|B^* f_k)|^2 = \sum_{k=1}^{\infty} \|B^* f_k\|^2 = N_{\mathcal{E}_B}(B^*)$$

where $\mathcal{E}_B$ is a basis for which we know $\sum_{k=1}^{\infty} \|Bf_k\|^2 < \infty$. To show the independence on a basis we start with $\mathcal{E} = \mathcal{E}_B$ which implies $N_{\mathcal{E}_B}(B) = N_{\mathcal{E}_B}(B^*) < \infty$, i.e. $B^* \in S_2$. Using the identity above we get $N_{\mathcal{E}}(B) = N_{\mathcal{E}_B}(B)$ for arbitrary basis $\mathcal{E}$.

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Exercise 27:
Show the following:

1. $S_2(H)$ is a two–sided $\star$–ideal in $\mathcal{L}(H)$ and $\|\cdot\|_{\nu_2}$ is a norm on it which fulfills
$$\|S\|_{\nu_2} \geq \|S\|_2$$
for all $S \in S_2(H)$. Recall that Hilbert–Schmidt operator is compact, i.e. $S_2(H) \subseteq \mathcal{K}(H)$.
2. $S_2(H)$ equipped with the inner product
$$(B|C)_2 = \sum_{j=1}^{\infty} (Be_j|Ce_j)_H$$
is a Hilbert space.

Proof

1. *Norm*: First we start by showing $\|\cdot\|_{\nu_2}$ is a norm. We note
$$\|(B+C)e_j\|^2 \leq \|Be_j\|^2 + 2\|Be_j\|\|Ce_j\| + \|Ce_j\|^2$$
and
$$(a+b)^s \leq a^s + b^s$$
for all $a, b \geq 0$ with $0 \leq s \leq 1$. This implies
$$\|B+C\|_{\nu_2} \leq \left(\sum_{j=1}^{\infty} (\|Be_j\| + \|Ce_j\|)^2 \right)^{\frac{1}{2}} \leq \|B\|_{\nu_2} + \|C\|_{\nu_2} .$$
It is obvious that
$$\|\alpha B\|_{\nu_2} = |\alpha| \|B\|_{\nu_2} .$$
Furthermore
$$\|B\|_{\nu_2} = 0 \Rightarrow \forall j : Be_j = 0$$
which implies that $B = 0$ by continuity. This proves that $(S_2, \|\cdot\|_{\nu_2})$ is a normed space.

Ideal: Let $D \in \mathcal{L}(H)$ and $B \in S_2(H)$ then we have
$$\|DBe_j\| \leq \|D\|\|Be_j\|$$
which gives $DB \in S_2$. From previous exercise we know that $C \in S_2(H)$ implies $C^* \in S_2(H)$. We have $D^*B^* \in S_2$ because adjoint of a bounded operator is bounded. It can be rewritten as $BD = (D^*B^*)^* \in S_2$.

Compactness: Consider a sequence of projectors P_n onto $\mathrm{lin}\{e_1, \ldots, e_n\}$ and define finite rank operators $B_n := P_n B$. Then
$$\|(B - B_n)x\|^2 = \sum_{j=n+1}^{\infty} |(e_j|Bx)|^2 \leq \|x\|^2 \sum_{j=n+1}^{\infty} \|B^* e_j\|^2 .$$
Using $B^* \in S_2$ we obtain $B_n \to B$. The limit implies that B is compact because it is a limit of finite rank operators.

2. *Scalar product*: The direct calculation shows that $\|\cdot\|_{\nu_2}$ satisfies parallelogram identity. Furthermore it is easy to check that $\|\cdot\|_{\nu_2}$ is generated by $(B,C)_{\nu_2}$.

   Completeness: Consider a ν_2-Cauchy sequence of Hilbert Schmidt operators B_n. Using the first part and the fact that $K(H)$ is a closed subspace of $\mathcal{L}(H)$ we obtain that B_n is also a Cauchy sequence in operator norm which converges to certain compact operator. We need to check that the limiting operator B is Hilbert Schmidt. For a given $\epsilon > 0$ and an orthonormal basis we have

$$\sum_{j=n+1}^{N} \|(B-B_n)e_j\|^2 \leq \|B-B_n\|_{\nu_2}^2 < \epsilon$$

   for large enough m, n and any $N = 1, 2, \ldots$. Taking first the limit $n \to \infty$ and then $N \to \infty$ we get $B \in S_2$ and $\|B - B_m\| < \epsilon$ for sufficiently large m, i.e. $\lim_{m\to\infty} \|B - B_m\| = 0$.

— **Turn the page!** —

Exercise 28:
Show the following:
Any trace–class operator S is Hilbert–Schmidt, and therefore compact, $S_1(H) \subseteq S_2(H) \subseteq \mathcal{K}(H)$ and

$$\mathrm{Tr}(|S|) \geq \|S\|_{\nu_2} \geq \|S\|_2 \,.$$

Proof
Consider $A \in S_1$ then

$$(\mathrm{Tr}|A|)^2 \geq \|A\|\mathrm{Tr}|A| \geq \mathrm{Tr}|A|^2 = \mathrm{Tr}A^*A = \|A\|_{\nu_2}^2 \,.$$

This implies $A \in S_1$ then also $|A|^2 \in S_1$ and $A \in S_2$.