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Spectral Theory

9th Exercise Sheet - Solutions

Exercise 29: *Polar decomposition for bounded operators*
Show that to any $B \in \mathcal{L}(H)$ there is just one partial isometry W_B such that $B = W_B|B|$ and $N(W_B) = N(B)$. Furthermore, the identity $R(W_B) = \overline{R(B)}$ holds.

Proof
We define

$$W_0 : R(|B|) \to R(B), W_0|B|x = Bx\,.$$

Due to the fact that $|B|$ might not be invertible we need to check that W_0 is well defined. This can be checked by showing $\|Bx\|^2 = \||B|x\|^2$ explicitly

$$\|Bx\|^2 = (Bx, Bx) = (x, B^*Bx) = (x, |B||B|x) = \||B|x\|^2\,.$$

i.e. W_0 preserves the norm. We construct W_B as the continuous extension of W_0. It maps $\overline{R(|B|)} = (\ker|B|)^\perp = (\ker B)^\perp$ to $\overline{R(B)}$. W_B is a partial isometry (W_B is an isometry on $(\ker W_B)^\perp$) which satisfies $W_B|B|x = Bx$ for all $x \in H$. We need to show uniqueness. Assume that there is an another partial isometry W with given properties. We have $(W_B - W)|B| = 0$. We use the orthogonal decomposition $x = y + z$ where $y \in \ker B$ and $z \in (\ker B)^\perp = \overline{R(|B|)}$. Using this decomposition we get

$$(W_B - W)x = (W_B - W)z = \lim_{n\to\infty}(W_B - W)|B|v_n = 0$$

where $|B|v_n \to z$. From the assumptions we have $(W_B - W)|B|v_n = 0$ which implies $(W_B - W)x = 0$ for every $x \in H$.

Exercise 30:
Show the following:
An operator $B \in \mathcal{L}(H)$ belongs to the trace class if and only if it is a product of two Hilbert–Schmidt operators.
Hint: Use the previous exercise.

Proof
Assume that $C, D \in S_2$. Then using polar decomposition we have $CD = W|CD|$ where W is a partial isometry. We prove two directions.
$\Leftarrow$: We use the following estimates

$$\mathrm{Tr}|CD| = \sum_{j=1}^{\infty} |(e_j, W^* CDe_j)| \leq \sum_{j=1}^{\infty} \|C^* We_j\| \|De_j\| \,.$$

Applying Hölder we get

$$\sum_{j=1}^{\infty} \|C^* We_j\| \|De_j\| \leq \left(\sum_{j=1}^{\infty} \|C^* We_j\|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} \|De_j\|^2\right)^{\frac{1}{2}} \,.$$

$\Rightarrow$: Let $B \in S_1$ then

$$\mathrm{Tr}|B| = \|\sqrt{|B|}\|_{\nu_2}^2 \,.$$

This implies that $\sqrt{|B|} \in S_2$. Furthermore $W\sqrt{|B|} \in S_2$ where W is a partial isometry corresponding to B. We can conclude that $B = (W\sqrt{|B|})\sqrt{|B|} = W|B| \in S_2$.

Exercise 31:
Let $W \in S_1(H)$. Then W is called a statistical operator if it is positive and fulfills the normalization condition $\mathrm{Tr}\, W = 1$. Show the following:

1. The set $\mathcal{W}$ of all statistical operators on a given H is convex.
2. An operator $W \in \mathcal{W}$ is one–dimensional projection if and only if it is an extremal point of $\mathcal{W}$, i.e., if and only if the condition $W = \alpha W_1 + (1-\alpha)W_2$ with $0 < \alpha < 1$ and $W_1, W_2 \in \mathcal{W}$ implies $W_1 = W_2 = W$.

Proof

1. Let W_1, W_2 be two statistical operators. For $\alpha \in [0,1]$ we have
$$\alpha W_1 + (1-\alpha)W_2 \geq 0 \,.$$
because $W_1 \geq 0$ and $W_2 \geq 0$. Furthermore $W_1 = |W_1|$ and $W_2 = |W_2|$ which implies
$$\mathrm{Tr}(\alpha W_1 + (1-\alpha)W_2) = \alpha \mathrm{Tr} W_1 + (1-\alpha)\mathrm{Tr} W_2 = \alpha + 1 - \alpha = 1 \,.$$
2. Assume $W^2 \neq W$. Then W has an eigenvalue $\alpha \in (0,1)$ associated to one dimensional projector E. We define a statistical operator $W' := \frac{1}{1-\lambda}(W - \lambda E)$ and we write
$$W = \lambda E + (1-\lambda)W' \,.$$
This implies that W is not an external point. On the other hand assume $W^2 = W$ and $W = \alpha W_1 + (1-\alpha)W_2$ for some $\alpha \in (0,1)$ where W_1 and W_2 are statistical operators. Then we have
$$\mathrm{Tr}(W^2) = \alpha^2 \mathrm{Tr} W_1^2 + \alpha(1-\alpha)\mathrm{Tr} W_1 W_2 + (1-\alpha)^2 \mathrm{Tr} W_2^2 = 1 \,.$$
Since $\mathrm{Tr}(W_1 W_2) \leq 1$ for any two statistical operators the only possibility how the equality above can be satisfied is $\mathrm{Tr} W_j^2 = 1$. This means that W_j has to be a one dimensional projections onto linear span of e_j s.t. $|(e_1|e_2)| = 1$.

Exercise 32:
Let $S \in L(H)$ be self-adjoint. Furthermore let $\Psi : B_b(\sigma(S)) \to \mathcal{L}(H)$ be the functional calculus from 13.6. Then for $\lambda \in \sigma(S)$ one has

$$\psi(1_{\{\lambda\}}) = 0 \Leftrightarrow \lambda \notin \sigma_p(S)\,.$$

More precisely, $\psi(1_{\{\lambda\}})$ is the orthogonal projection onto $N(\lambda - S)$.

Proof
To prove this Theorem we use the following Lemma

Lemma: Let H be a HIlbert space and $S \in \mathcal{L}(H)$ be a self-adjoint operator and $\lambda \in \sigma_p(S)$. Then

$$\psi(f)x = f(\lambda)x \Leftrightarrow \lambda \notin \sigma_p(S)\,, \quad \forall f \in B_b(\sigma(S)), x \in E_\lambda := \ker(\lambda - S)\,.$$

Sketch of Proof: Inductively. We show it for monomials, then polynomials, by denseness it translates to continuous functions and as a final step to $f \in B_b(\sigma(S))$. For the monomial m of order n we have

$$\psi(m)x = m(S)x = S^n x = \lambda^n x = m(\lambda)x\,.$$

First we show that $P := \psi(1_{\{\lambda_0\}})$ is an orthogonal projector. This can be easily checked by using properties of functional calculus. The function $1_{\{\lambda_0\}}$ is positive which implies P is self-adjoint. The function $1^2_{\{\lambda_0\}} = 1_{\{\lambda_0\}}$ implies $P^2 = P$. The self-adjoint operator satisfying $P^2 = P$ is orthogonal projector. Furthermore we have $R(P) \subseteq N(\lambda - S)$ by

$$SP = \psi(p_1)\psi(1_{\{\lambda_0\}}) = \lambda P\,.$$

Last we use the following equivalences

$$\begin{aligned}\lambda &\Leftrightarrow \exists x \in H \setminus \{0\} : Sx = \lambda x \\ &\Leftrightarrow \exists x \in H \setminus \{0\} : \psi(1_{\{\lambda_0\}})x = 1_{\{\lambda\}}(\lambda_0)x \neq 0 \\ &\Leftrightarrow \exists z \in H : \psi(1_{\{\lambda_0\}})z \neq 0\end{aligned}$$

The last line is equivalent to

$$\lambda_0 \notin \sigma_p(S) \Leftrightarrow \psi\psi(1_{\{\lambda_0\}})z = 0$$

on H.