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Spectral Theory

10th ExerciseSheet - Solutions

Exercise 33:
Let A be a self-adjoint positive operator. Then we define for $t \in \mathbb{R}^+$

$$T(t) := e^{-tA} .$$

Show that

1. $T(t)$ is self-adjoint.
2. $T(t)$ is a strongly continuous semigroup, i.e. $T(t)T(s) = T(t+s)$ for every $s, t \in \mathbb{R}^+$.
3. Furthermore let $x \in \mathcal{D}(A)$ and $Ax = y$. Then for every $s > 0$

$$\lim_{t \to 0_+} \frac{1}{t}(T(s)x - T(t+s)x) = T(s)y$$

   holds.

Proof
Using Theorem 13.14 we have

$$H = \oplus_{n\in\mathbb{N}} H_n \quad \{x_k | x_k \in H_n \, \forall n \sum_{k\in\mathbb{N}} \|x_k\|^2 < \infty\}$$

and

$$Ax = \sum_{n\in\mathbb{N}} AP_n x \, , \quad \forall x \in \mathcal{D}(A) = \{x \in H | \sum_{n\in\mathbb{N}} \|AP_n x\|^2 < \infty\} \, .$$

Furthermore using construction from Theorem 13.15 we obtain

$$f(A)x := \sum_{n\in\mathbb{N}} f(A_n) P_n x$$

for each $x \in H$. From Theorem 13.15 we have $\psi(f)^* = \psi(\overline{f})$. One can check $\bar{(}f(x)) = \exp(-tx) = f(x)$. This means that T is self-adjoint. Next

$$T(t)T(s) = \psi(\exp(-t\cdot))\psi(\exp(-s\cdot)) = \psi(\exp(-(t+s)\cdot)) = T(t+s) \, .$$

Last we have

$$\frac{1}{t}(T(s)x - T(t+s)x) = \frac{1}{t}\left(T(s)x - T(t+s)x\right) = \frac{1}{t}\left(\psi(\exp(-s\cdot)x - \psi(\exp(-(t+s)\cdot)x\right) ,$$
$$\frac{1}{t}\left(\psi(\exp(-s\cdot) - \psi(\exp(-(t+s)\cdot))\right) = \frac{1}{t}\left(1 - \psi(\exp(-t\cdot)\right)\psi(\exp(-s\cdot) \to \psi(\exp(-s\cdot)\cdot = T(s)A$$

which implies desires claim.

Exercise 34:
Let $T \in \mathcal{L}(H)$ be self-adjoint. Show or calculate the following:

1. Let $\sigma(T) = \{0, 1\}$, then T is orthogonal projector,
2. $\|TR(\lambda, T)^2\|_{\mathcal{L}(H)}$ for $\lambda \in \rho(T)$,
3. $\|Te^{-sT^2}\|_{\mathcal{L}(H)}$ for $s \geq 0$,
4. Assume that T is positive $\|(I + sT)^{-1}e^{isT}\|_{\mathcal{L}(H)}$ for $s \geq 0$.

Proof
In each of the claims we use $\|\psi(T)\| = \|\psi\|_\infty$.

1. We start by
$$\|T^2 - T\| = \|\psi(\cdot^2 - \cdot)\| = \|x^2 - x\|_\infty = \sup_{\lambda \in \{0,1\}} |\lambda - 1||\lambda| = 0 \,.$$
This implies $T^2 = T$, i.e. T is a projection. Assume $z \in R(T)^\perp \subseteq H$ Then
$$(Tz, x) = (z, T^*x) = (z, Tx) = 0$$
for all $x \in H$ where we used that T is self-adjoint. For $x = Tz$ we have $\|Tz\|_H = \sqrt{(Tz, Tz)} = 0$ which implies $Tz = 0$, i.e. $z \in N(T)$. We conclude $H = R(T) \oplus N(T)$, i.e. T is orthogonal projector onto $R(T)$.
2. For $\lambda \in \rho(T)$ we define four elements $z_\pm^\pm \in \sigma(T)$ s.t.
$$\begin{aligned}
\left|\lambda - z_+^+\right| &= d\left(\frac{1}{\sqrt{2s}}, \sigma(T) \cap (\lambda, \infty)\right) \,, \\
\left|\lambda - z_-^+\right| &= d\left(\frac{1}{\sqrt{2s}}, \sigma(T) \cap (-\infty, \lambda)\right) \,, \\
\left|-\lambda - z_+^-\right| &= d\left(-\frac{1}{\sqrt{2s}}, \sigma(T) \cap (-\lambda, \infty)\right) \,, \\
\left|-\lambda - z_-^-\right| &= d\left(-\frac{1}{\sqrt{2s}}, \sigma(T) \cap (-\infty, -\lambda)\right) \,.
\end{aligned}$$
We distinguish three cases.
$\lambda = 0$: We have
$$\|TR(0, T^2)\| = \| - R(0, T)\| = \|T^{-1}\| = \frac{1}{d(0, \sigma(T))}$$
$\lambda \neq 0$: We consider a function $f : \mathbb{R} \setminus \{\lambda\} \to \mathbb{R}$, $\frac{x}{(\lambda - x)^2}$. Its derivative is
$$f'(x) = \frac{\lambda + x}{(\lambda - x)^3} \,.$$
for all $x \in \mathbb{R} \setminus \{\lambda\}$. One can check that the function has a local maximum at $x_0 = -\lambda$. It goes to infinity as $x \to \lambda$ and it vanishes for $x \to \infty$.
$\lambda \neq 0$, $-\lambda \notin \rho(T)$: We have
$$\|\psi(f)\| = \sup_{x \in \sigma(T)} |f(x)| = \max_{x \in \{-\lambda, z_+^-, z_+^+\}} |f(x)| \,.$$
$\lambda \neq 0$, $-\lambda \in \rho(T)$: We have
$$\|\psi(f)\| = \sup_{x \in \sigma(T)} |f(x)| = \max_{x \in \{z_-^-, z_-^+, z_+^-, z_+^+\}} |f(x)| \,.$$

3. For $s>0$ we set four elements $z_\pm^\pm \in \sigma(T)$ s.t.

$$\begin{aligned}
\left|\frac{1}{\sqrt{2s}} - z_+^+\right| &= d\left(\frac{1}{\sqrt{2s}}, \sigma(T) \cap \left(\frac{1}{\sqrt{2s}}, \infty\right)\right), \\
\left|\frac{1}{\sqrt{2s}} - z_-^+\right| &= d\left(\frac{1}{\sqrt{2s}}, \sigma(T) \cap \left(-\infty, \frac{1}{\sqrt{2s}}\right)\right), \\
\left|-\frac{1}{\sqrt{2s}} - z_+^-\right| &= d\left(-\frac{1}{\sqrt{2s}}, \sigma(T) \cap \left(-\frac{1}{\sqrt{2s}}, \infty\right)\right), \\
\left|-\frac{1}{\sqrt{2s}} - z_-^-\right| &= d\left(-\frac{1}{\sqrt{2s}}, \sigma(T) \cap \left(-\infty, -\frac{1}{\sqrt{2s}}\right)\right).
\end{aligned}$$

We distinguish between several cases.
$s=0$: We have $Te^{-sT^2} = T$. Using functional calculus we get

$$\|Te^{-0T^2}\| = \|T\| = \sup_{z\in\sigma(T)} |z| = r(T).$$

$s>0$: We set $f:\mathbb{R}\to\mathbb{R}, xe^{-sx^2}$. This function is differentiable with the derivative

$$f'(x) = (1-2sx^2)e^{-sx^2} = 2s\left(\frac{1}{\sqrt{2s}} - x\right)\left(\frac{1}{\sqrt{2s}} + x\right)e^{-sx^2}.$$

One can check that

$$f'(x) = 0 \Leftrightarrow x = \pm\frac{1}{\sqrt{2s}}.$$

Furthermore

$$\lim_{x\to\pm\infty} |f(x)| = 0.$$

We can conclude that $|f|$ has a global Maximum $|f(x_0)| = \frac{1}{\sqrt{2s}}e^{-\frac{1}{2}}$ for $x_0 = \pm\frac{1}{\sqrt{2s}}$. Then we have

$$\begin{aligned}
\frac{1}{\sqrt{2s}} \in \sigma(T) \text{ or } -\frac{1}{\sqrt{2s}} \in \sigma(T) &: \|Te^{-sT^2}\| = \sup_{x\in\sigma(T)} |f(x)| = e^{-\frac{1}{2}} \\
\pm\frac{1}{\sqrt{2s}} \in \rho(T) &: \|Te^{-sT^2}\| = \sup_{x\in\sigma(T)} |f(x)| = \max_{x\in\{z_-^-, z_-^+, z_+^-, z_+^+\}} |f(x)|.
\end{aligned}$$

4. We distinguish between two cases.
$s=0$: We have

$$(I+sT)^{-1}e^{isT} = I^{-1} = I.$$

which means

$$\|(I+sT)^{-1}e^{isT}\| = \|I\| = 1.$$

$s>0$: Due to $\sigma(T) \subseteq (0,\infty)$ we know $-\frac{1}{s} \in \rho(T)$. We write

$$(I+sT)^{-1}e^{isT} = -\frac{1}{s}\left(-\frac{1}{s} - T\right)^{-1} e^{isT} = -\frac{1}{s}R\left(-\frac{1}{s}, T\right)e^{isT}.$$

We investigate a function $f:\mathbb{R}\setminus\left\{-\frac{1}{s}\right\} \to \mathbb{C}, -\frac{1}{s}\frac{1}{-\frac{1}{s}-x}e^{isx}$. This function is continuous on $\mathbb{R}\setminus\left\{-\frac{1}{s}\right\}$. Its absolute value is

$$|f| = \frac{1}{s}\left|\frac{1}{-\frac{1}{s} - x}\right|.$$

for all $x \in \mathbb{R}\setminus\left\{-\frac{1}{s}\right\}$. We know from functional calculus $\psi(f) = f(T)$. Then

$$\|\psi(f)\| = \sup_{x\in\sigma(T)} |f(x)| = \frac{1}{s}\sup_{x\in\sigma(T)}\left|\frac{1}{-\frac{1}{s}-x}\right| = \frac{1}{s}\frac{1}{\inf_{x\in\sigma(T)}\left|-\frac{1}{s}-x\right|} = \frac{1}{s}\frac{1}{d\left(-\frac{1}{s}, \sigma(T)\right)}.$$

Exercise 35:
Let $A, B \in \mathcal{L}(H)$. Then

$$e^{A+B} = \lim_{N\to\infty} \left(e^{A/N} e^{B/N}\right)^N .$$

Proof
We define

$$S_N := \exp\left(\frac{A+B}{N}\right) = 1 + \frac{A+B}{N} + \frac{A^2 + AB + BA + B^2}{2N^2} + \mathcal{O}(N^{-3}),$$
$$T_N := \exp\left(\frac{A}{N}\right)\exp\left(\frac{B}{N}\right) = \left(1 + \frac{A}{N} + \frac{A^2}{2N^2} + \mathcal{O}(N^{-3})\right)\left(1 + \frac{B}{N} + \frac{B^2}{2N^2} + \mathcal{O}(N^{-3})\right) .$$

By a direct calculation we have

$$S_N - T_N = \frac{BA - AB}{2N^2} + \mathcal{O}(N^{-3})$$

i.e.

$$\|S_N - T_N\| \leq \frac{C}{N^2}$$

for certain $C < \infty$. Writing telescopic sum of $S_N^N - T_N^N$ we acquire

$$S_N^N - T_N^N = \sum_{j=0}^{N-1} S_N^j (S_N - T_N) T_N^{N-1-j}$$

estimating the sum using properties of functional calculus ($\|\psi(A)\| = \|1_{\sigma(A)}\psi\|_\infty$) we have

$$\|S_N^N - T_N^N\| \leq N\|S_N - T_N\| \left(\max\{\|S_N\|, \|T_N\|\}\right)^N \leq \frac{C}{N}\exp(\|A\| + \|B\|) .$$

We conclude

$$\lim_{N\to\infty} \left(\exp\left(\frac{A}{N}\right) + \exp\left(\frac{B}{N}\right)\right)^N = \exp(A+B) .$$