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Spectral Theory

11th Exercise Sheet - Solutions

Exercise 36:
Let A be a self-adjoint operator in H. Let $f, g : \sigma(A) \to \mathbb{C}$ Borel measurable and locally bounded. Consider H_n, P_n and A_n be defined as in Thm 13.15. Let

$$f(A)x := \sum_{n\in\mathbb{N}} f(A_n)P_n x$$

for every $x \in \mathcal{D}(f(A))$ where $D(f(A)) = \{x \in X | \sum_{n\in\mathbb{N}} \|f(A_n)P_n x\|^2 < \infty\}$. Show that

1. $f(A) + g(A) \subseteq (f + g)(A)$,
2. $f(A)g(A) \subseteq (fg)(A)$.

Proof

1. Let $x \in H$ s.t. $f(A)x = \sum_{n\in\mathbb{N}} f(A_n)P_n x$, $\sum_{n\in\mathbb{N}} \|f(A_n)P_n x\|^2 < \infty$ and $g(A)x = \sum_{n\in\mathbb{N}} g(A_n)P_n x$, $\sum_{n\in\mathbb{N}} \|g(A_n)P_n x\|^2 < \infty$. Then

$$\begin{aligned}\sum_{n\in\mathbb{N}} \|[f(A_n) + g(A_n)]P_n x\|^2 &\leq \sum_{n\in\mathbb{N}} \left(\|f(A_n)P_n x\|^2 + \|g(A_n)P_n x\|^2\right) \\ &\leq \sum_{n\in\mathbb{N}} \|f(A_n)P_n x\|^2 + \sum_{n\in\mathbb{N}} \|g(A_n)P_n x\|^2\end{aligned}$$

and $(f(A) + g(A))x = \sum_{n\in\mathbb{N}}[f(A_n) + g(A_n)]P_n x$.

2. We have

$$f(A)g(A)x = f(A)\sum_{n\in\mathbb{N}} g(A_n)P_n x\,.$$

and

$$(fg)(A)x = \sum_{n\in\mathbb{N}}(fg)(A_n)P_n x = \sum_{n\in\mathbb{N}} f(A_n)g(A_n)P_n x = \sum_{n\in\mathbb{N}} f(A_n)P_n g(A)x = f(A)(g(A)x)$$

for x s.t. $\sum \|(fg)(A_n)P_n x\|^2 < \infty$ and $\sum \|g(A_n)P_n x\|^2 < \infty$. This means that for

$$x \in \mathcal{D}(f(A)g(A)) \subseteq \mathcal{D}(g(A))$$

we have $\sum \|(fg)(A_n)P_n x\|^2 = \sum \|f(A_n)P_n g(A)x\|^2 < \infty$. We can conclude $\mathcal{D}(f(A)g(A)) = \mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A))$.

Exercise 37:
Let $a \in \mathbb{R}^{d\times d}$ be symmetric such that $\xi^t a\xi \geq \eta\|\xi\|^2$ for certain $\eta > 0$ for every $\xi \in \mathbb{R}^d$ and $b \in \mathbb{R}^d$. We define

$$\mathfrak{a}(u,v) := \int \overline{\nabla v}^t a\nabla u + (b^t\nabla u)\overline{v}\,.$$

Show that

$$|\operatorname{Im}\mathfrak{a}(u,u)| \leq |b|\|\nabla u\|_2\|u\|_2 \leq \epsilon^2\|\nabla u\|_2^2 + \frac{|b|}{2\epsilon^2}\|u\|_2^2 \leq \frac{\epsilon^2}{4}\operatorname{Re}\mathfrak{a}(u,u) + \frac{|b|}{2\epsilon^2}\|u\|_2^2\,.$$

Notice that even for the case that a is real and symmetric it is possible that $|\operatorname{Im}\mathfrak{a}(u,u)| \neq 0$.

Proof
We start by rewriting the form in Fourier picture. We know $\nabla u = 2\pi i\xi\hat{u}$. This implies

$$a(u,u) = \int_{\mathbb{R}^d}(4\pi^2\xi^t a\xi + 2\pi i b^t\xi)|\hat{u}(\xi)|^2\mathrm{d}\xi\,.$$

We denote $a(\xi) = 4\pi^2\xi^t a\xi + 2\pi i b^t\xi$. It is easy to see that imaginary part of $a(\xi)$ corresponds to $2\pi i b^t\xi$. Next using the following estimates

$$\begin{aligned}&|\operatorname{Im}a(\xi)| \leq 2\pi|b^t\xi| \leq 2\pi|b||\xi|\\&4\pi^2\eta|\xi|^2 \leq 4\pi^2\,.\end{aligned}$$

we conclude

$$\begin{aligned}&|\xi| \leq \frac{1}{2\pi\sqrt{\eta}}\sqrt{4\pi^2\xi^t a\xi}\\&|\operatorname{Im}a(\xi)| \leq \frac{|b|}{\eta}\sqrt{|\operatorname{Re}a(\xi)|}\,.\end{aligned}$$

To show explicitly the claim in the theorem we write

$$\operatorname{Im}a(u,u) = \frac{a(u,u) - \overline{a(u,u)}}{2i} = \frac{\int_{\mathbb{R}^d}(b^t\nabla u)\overline{u} - (b^t\nabla\overline{u})u\mathrm{d}x}{2i} = \frac{\int_{\mathbb{R}^d}(b^t\nabla u)\overline{u}\mathrm{d}x}{i}\,.$$

The rest is simple application of Cauchy-Schwarz.

Exercise 38:
Find the adjoint operator and deficiency indices of it for the following cases:

1. $S_1 = -\Delta$, $\mathcal{D}(S) := W^{2,2}(\mathbb{R})$,
2. $S_2 = -i\frac{\partial}{\partial x}$, $\mathcal{D}(S) := W^{1,2}(\mathbb{R}^+)$,
3. $S_3 = -i\frac{\partial}{\partial x}$, $\mathcal{D}(S) := W_0^{1,2}(\mathbb{R}^+)$.

Proof

1. We start by finding the adjoint operator. By a direct calculation for $\varphi \in C_c^\infty(\mathbb{R})$ we have

$$(\varphi, S_1\psi) = -(\varphi, \psi'') = (\varphi', \psi') - \left[\varphi\psi'\right]_{-\infty}^{\infty} = -(\varphi'', \psi)$$

where we used that $\psi \in W^{2,2}(\mathbb{R})$. By density argument we can extend the above to $\varphi \in W^{2,2}(\mathbb{R})$ which means that S_1 is self-adjoint and $n_+(S_1) = n_-(S_1) = 0$.

2. Similarly as in the first case we write for $\varphi \in C_c^\infty(\mathbb{R}^+)$

$$(\varphi, S_2\psi') = -i(\varphi, \psi') = i(\varphi', \psi) - i\left[\varphi\psi\right]_0^\infty = (-i\varphi', \psi)$$

where we used $\varphi(0) = 0$ and $\psi \in W^{1,2}(\mathbb{R}^+)$. Again we can extend the above equalities up to $\varphi \in W_0^{1,2}(\mathbb{R}^+)$. We can see that $S_2^* = S_3$. This means that the operator S_2 is not symmetric because $W_0^{1,2}(\mathbb{R}^+) \subset W^{1,2}(\mathbb{R}^+)$.

3. From the previous we know that $S_3^* = S_2$. This means that the operator is symmetric. We calculate the solutions of

$$S_3^*\psi_- = i\psi_-$$
$$S_3^*\psi_+ = -i\psi_+$$

Direct calculation shows that $\psi_\pm = \exp(\pm x)$. We have $\psi_- \in\subset W^{1,2}(\mathbb{R}^+)$ and $\psi_+ \notin\subset W^{1,2}(\mathbb{R}^+)$. We conclude $n_+(S_3) = \dim(N(S_3^*+i)) = 0$ and $n_-(S_3) = \dim(N(S_3^*-i)) = 1$.

Exercise 39:
Suppose D is a vector space. Let $s(f,g)$ be a sesquilinear form on D and $q(f) = s(f,f)$ the associated quadratic form. Prove the parallelogram law

$$q(f+g) + q(f-g) = 2q(f) + 2q(g)\,.$$

and the polarization identity

$$s(f,g) = \frac{1}{4}[q(f+g) - q(f-g)] + \frac{i}{4}[q(f-ig) - q(f+ig)]\,.$$

Show that $s(f,g)$ is symmetric if and only if $q(f)$ is real-valued.

Proof
The parallelogram law is proved by direct calculation

$$\begin{aligned} q(f+g) + q(f-g) &= s(f+g, f+g) + s(f-g, f-g) \\ &= s(f,f) + s(g,g) + s(f,g) + s(g,f) + s(f,f) + s(g,g) - s(f,g) - s(g,f) \\ &= 2q(f) + 2q(g)\,. \end{aligned}$$

Similarly one shows polarization identity. For $|\xi| = 1$ we start with

$$\begin{aligned} q(\overline{\xi}f+g) &= s(\overline{\xi}f+g, \overline{\xi}f+g) = |\xi|^2 s(\overline{\xi}f+g, \overline{\xi}f+g) \\ &= \xi\overline{\xi}s(\overline{\xi}f+g, \overline{\xi}f+g) = s(|\xi|^2 f + \xi g, |\xi|^2 f + \xi g) \quad = s(f+\xi g, f+\xi g) = q(f+\xi g)\,. \end{aligned}$$

which implies

$$\frac{1}{4}[q(f+g) - q(f-g)] + \frac{i}{4}[q(f-ig) - q(f+ig)] = \frac{1}{4}\sum_{\xi^4=1} \xi q(\xi f+g)\,.$$

Furthermore for $\xi \in \mathbb{C}$, $|\xi| = 1$ holds

$$\begin{aligned} \xi q(\xi f+g) &= s(\xi f+g, \xi f+g) \\ &= \xi\left(|\xi|^2 s(f,f) + s(g,g) + \xi s(f,g) + \overline{\xi}s(g,f)\right) \\ &= \xi(q(f) + q(g)) + s(f,g) + \xi^2 s(g,f)\,. \end{aligned}$$

Since $\sum_{\xi^4=1} \xi = 0$ and $\sum_{\xi^4=1} \xi^2 = 0$ we have $s(f,g) = \frac{1}{4}\sum_{\xi^4=1} \xi q(\xi f+g)$. For the last part we part we show two implications. Assume that s is symmetric. Then

$$q(f) = s(f,f) = \overline{s(f,f)} = \overline{q(f)}\,.$$

for all $f \in D$. This means $q(f) \in \mathbb{R}$.
Conversely assume that $q(f) \in \mathbb{R}$. Then by polarization identity

$$s(f,g) = \frac{1}{4}\sum_{\xi^4=1} \xi q(\xi f+g) = \frac{1}{4}\sum_{\overline{\xi}^4=1} \overline{\xi} q(\overline{\xi}f+g) = \frac{1}{4}\sum_{\overline{\xi}^4=1} \overline{\overline{\xi} q(\overline{\xi}f+g)} = \frac{1}{4}\sum_{\overline{\xi}^4=1} \overline{\xi q(f+\xi g)} = \overline{s(f,g)}\,.$$