Spectral Theory
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These lecture notes contain a summary of the course. They are intended to be used parallel to the lecture. They are **not** meant to be used as a text book or for self studies. Attending the lectures cannot be substituted by reading these lecture notes.

The lecture itself is a natural continuation of the Functional Anylsis lecture in Winter 2018/19. Sections are numbered consecutively.

8 The spectrum of closed operators

Notation: i) When nothing else is said, X, Y, and Z are complex Banach spaces.

ii) $\mathcal{L}(X,Y) := \{T : X \to Y : \ T \text{ is linear and bounded}\}$. For $T \in \mathcal{L}(X,Y)$ we have the *operator norm*

$$\|T\| := \sup_{\|x\|_X \leq 1} \|Tx\|_Y,$$

and $(\mathcal{L}(X,Y), \|\cdot\|)$ is a Banach space. For the following we also refer to FA[1] 2.14.

8.1. Definition: A *linear operator* A from X to Y is a linear operator $A : D(A) \to Y$ where $D(A)$, the *domain* of A, is a linear subspace of X. We also write $A : X \supseteq D(A) \to Y$. Further we use

$$\begin{aligned} R(A) &:= \{Ax : x \in D(A)\} \text{ *range* of } A, \\ N(A) &:= \{x \in D(A) : Ax = 0\} \text{ *kernel* (*null space*) of } A. \end{aligned}$$

A linear operator A from X to Y is called *closed* if its graph

$$\operatorname{gr}(A) := \{(x, Ax) : x \in D(A)\} \subseteq X \times Y$$

is a closed subspace of $X \times Y$.

Recall: Since A is linear, its graph $\operatorname{gr}(A)$ is a linear subspace of $X \times Y$. The space $X \times Y$, equipped with the norm given by $\|(x,y)\| := \|x\|_X + \|y\|_Y$ is a Banach space.

8.2. Lemma: Let $A : X \supseteq D(A) \to Y$ be a linear operator. The following are equivalent:

(i) A is closed,

(ii) $D(A)$ is a Banach space for the *graph norm* given by

$$\|x\|_A := \|x\|_X + \|Ax\|_Y.$$

(iii) For every sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A)$ and $x \in X$, $y \in Y$ such that $x_n \to x$ in X and $Ax_n \to y$ in Y one has $x \in D(A)$ and $Ax = y$.

Notation: We write $[D(A)] := (D(A), \|\cdot\|_A)$. Then obviously $A \in \mathcal{L}([D(A)], Y)$.

Proof. By definition, closedness of A is equivalent to closedness of $\operatorname{gr}(A)$, and since $X \times Y$ is a Banach space, this is equivalent to completeness of $\operatorname{gr}(A)$. Since the map

$$J : [D(A)] \to \operatorname{gr}(A), x \mapsto (x, Ax),$$

is linear, bijective, and isometric, the latter is equivalent to $[D(A)]$ being a Banach space. On the other hand, closedness of $\operatorname{gr}(A)$ means

[1] FA means the Functional Analysis lecture in winter 2018/19.

for any sequence (x_n, Ax_n) in $\mathrm{gr}(A)$ and $(x,y) \in X \times Y$ such that $(x_n, Ax_n) \to (x,y)$ one has $(x,y) \in \mathrm{gr}(A)$,

which clearly is equivalent to (iii). $\square$

Examples: 1) Let $I \subseteq \mathbb{R}$ an open interval, $p \in [1,\infty]$, $X = Y = L^p(I)$, $A := \frac{d}{dx}$ with $D(A) = W^{1,p}(I) = \{f \in L^p(I) : f' \in L^p(I)\}$ (weak derivative). Then A is closed. Recall first ($\to$ FA 5.1):

$f \in L^1_{\mathrm{loc}}(I)$ has *a weak derivative in* I if there exists $g \in L^1_{\mathrm{loc}}(I)$ such that, for any $\varphi \in C_c^\infty(I)$, we have

$$-\int_I f\varphi'\,dx = \int_I g\varphi\,dx.$$

We show that A is closed: Let (f_n) be a sequence in $W^{1,p}(I)$ with $f_n \to f$ and $f_n' \to g$ with respect to $\|\cdot\|_p$. Then for $\varphi \in C_c^\infty(I) \subseteq L^{p'}(I)$ (where $\frac{1}{p} + \frac{1}{p'} = 1$) we have, by Hölder, $f_n\varphi' \to f\varphi'$ and $g_n\varphi \to g\varphi$ in $\|\cdot\|_1$. This implies

$$-\int_I f\varphi'\,dx = \lim_n -\int_I f_n\varphi'\,dx = \lim_n \int_I g_n\varphi\,dx = \int_I g\varphi\,dx.$$

Hence f has weak derivative $g \in L^p(I)$, so $f \in W^{1,p}(I)$ and $f' = g$.

2) Let $X = Y = L^1(0,1)$, $A := \frac{d}{dx}$ with $D(A) = C^1[0,1]$. Then A is *not* closed: Take $g \in L^1(0,1) \setminus C[0,1]$, say $g = 1_{[1/2,1]} - 1_{[0,1/2)}$, and approximate g in $\|\cdot\|_1$ by a sequence (g_n) in $C[0,1]$. Then let

$$f(t) := \left|t - \frac{1}{2}\right| \quad \text{and} \quad f_n(t) := \frac{1}{2} + \int_0^t g_n(s)\,ds.$$

We obtain $f_n \in C^1[0,1]$ and $f_n' = g_n \to g$ in $\|\cdot\|_1$, $f_n \to f$ in $\|\cdot\|_\infty$, hence in particular $f_n \to f$ in $\|\cdot\|_1$. But $f \notin D(A)$. [As a concrete g_n one can take

$$g_n(t) := \begin{cases} -1 & , t \in [0, \frac{1}{2} - \frac{1}{n}) \\ n(t - \frac{1}{2} & , |t - \frac{1}{2}| \le \frac{1}{n} \\ 1 & , t \in (\frac{1}{2} + \frac{1}{n}, 1] \end{cases} \quad , \quad n \ge 3.]$$

3) Let $X = L^1(0,1)$, $Y = \mathbb{C}$, $Af := f(0)$ with $D(A) = C[0,1]$. Then A is not closed: Take $f_n(t) := (1-nt)1_{[0,1/n]}(t)$. Then $f_n \to 0 =: f$ in $\|\cdot\|_1$ and $Af_n = f_n(0) = 1 \to 1$. Here $f \in D(A)$, but $Af = f(0) = 0 \ne 1$.

Comment: $L^1[0,1]$ is a Banach space whose elements are equivalence classes of functions that coincide almost everywhere (a.e.). Strictly speaking, $C[0,1]$ should be considered here as the set of equivalence classes in $L^1[0,1]$ that contain a continuous function. This continuous function is then unique within its class. A evaluates the unique continuous function in the class at 0. In fact, a similar interpretation has to be given to $C^1[0,1]$ as a subspace of $L^1[0,1]$ in Example 2) above.

4) Let $p \in [1,\infty]$, $X = Y = L^p(\Omega,\mu)$ where (Ω,μ) is a σ-finite measure space. Let $m : \Omega \to \mathbb{C}$ be measureable and

$$Af := mf \text{ with } D(A) := \{f \in L^p(\Omega,\mu) : mf \in L^p(\Omega,\mu)\}.$$

Then A is closed: If $f_n \to f$ and $mf_n \to g$ with respect to $\|\cdot\|_p$, then we find subsequences such that $f_{k(n)} \to f$ and $mf_{k(l(n))} \to g$ almost everywhere. Hence $mf = g$ a.e., and by $g \in L^p$ we have $f \in D(A)$, $Af = g$.

We also recall the following (cp. FA 2.15).

8.3. Closed Graph Theorem: Let $A : X \to Y$ be a linear operator. Then

$$A \text{ is closed} \quad \Longleftrightarrow \quad A \text{ is bounded.}$$

Notation: Let A and B be linear operators form X to Y and C be a linear operator from Y to Z. We define

$$\begin{aligned} A+B \quad &\text{by} \quad (A+B)x := Ax + Bx \text{ for } x \in D(A+B) := D(A) \cap D(B), \\ CA \quad &\text{by} \quad (CA)x := C(Ax) \text{ for } x \in D(CA) := \{\tilde{x} \in D(A) : A\tilde{x} \in D(C)\}. \end{aligned}$$

$A+B$ is a linear operator from X to Y, and CA is a linear operator from X to Z.

Sum and product of linear operators are associative, but not distributive in general.

In general, sums or products of closed operators are not closed.

Example: Take $X = l^1$ and, for $x = (x_n)$ define A by $(Ax)_n = nx_{n-1}$ if n is even and $(Ax)_n = 0$ if n is odd, and let $D(A) := \{x \in l^1 : Ax \in l^1\}$.

Then A is closed: If $x^{(n)}$ is a sequence in $D(A)$ such that $x^{(n)} \to x$ and $Ax^{(n)} \to y$, then $x_k = \lim_n x_k^{(n)}$ for each k, and $y_k = \lim_n kx_{k-1}^{(n)}$ for even k, $y_k = 0$ for odd k. We conclude that $y_k = kx_{k-1}$ for even k. Hence we have $Ax = y$, and $x \in D(A)$ by $y \in l^1$.

However, $B := A + (-A) = 0$ with $D(B) = D(A)$ and $C := AA = 0$ with $D(C) = D(A)$ are not closed (otherwise $D(A)$ would be a closed subspace of l^1, but $D(A) \neq l^1$ is dense in l^1).

8.4. Lemma (Properties of closed operators): Let A be a closed linear operator from X to Y, $T \in \mathcal{L}(X,Y)$, and $S \in \mathcal{L}(Z,X)$. Then:

(a) $B := A + T$ is closed (here $D(B) = D(A)$).

(b) $C := AS$ is closed (here $D(C) = \{z \in Z : Sz \in D(A)\}$).

(c) If A is injective then A^{-1} is closed (here $D(A^{-1}) = R(A)$).

(d) If R is injective and closed from Y to Z such that $R^{-1} \in \mathcal{L}(Z, Y)$ then $D := RA$ is closed (here $D(D) = \{x \in D(A) : Ax \in D(R)\}$).

Proof. (a) Take (x_n) in $D(A)$ such that $x_n \to x$, $Bx_n \to y$. Since T is bounded we have $Tx_n \to Tx$. Now $(A+T)x_n \to y$ implies that $Ax_n \to y - Tx$. By closedness of A we obtain $x \in D(A)$ and $Ax = y - Tx$, i.e. $Bx = y$. end Tue 23.04.19

(b) Let (z_n) be a sequence in $D(C)$ such that $z_n \to z$, $Cz_n \to y$. Since S is bounded we have $Sz_n \to Sz$, $A(Sz_n) \to y$. But (Sz_n) is a sequence in $D(A)$, thus $Sz \in D(A)$, $A(Sz) = y$ by closedness of A. We have shown $z \in D(C)$, $Cz = y$.

(c) follows from $\mathrm{gr}(A^{-1}) = \{(y, x) : (x, y) \in \mathrm{gr}(A)\}$.

(d) Let (x_n) be a sequence in $D(D)$ such that $x_n \to x$ and $Dx_n \to z$. By $R^{-1} \in \mathcal{L}(Z, Y)$ we have $Ax_n = R^{-1}Dx_n \to R^{-1}z$. Since A is closed and $D(D) \subset D(A)$, we obtain $x \in D(A)$ and $Ax = R^{-1}z \in R(R^{-1}) = D(R)$. Hence $x \in D(D)$ and $Dx = z$. □

8.5. Definition (spectrum and resolvent): Let $A : X \supseteq D(A) \to X$ be a linear operator. Then

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda I - A : D(A) \to X \text{ is bijective and } (\lambda I - A)^{-1} \in \mathcal{L}(X)\}$$

is called the *resolvent set* of A and

$$\sigma(A) := \mathbb{C} \setminus \rho(A)$$

is the *spectrum* of A. The map

$$\rho(A) \to \mathcal{L}(X), \quad \lambda \mapsto (\lambda I - A)^{-1},$$

is called the *resolvent* of A, and for $\lambda \in \rho(A)$, the operator

$$R(\lambda, A) := (\lambda I - A)^{-1}$$

is called the *resolvent operator* (at λ).

Remark: It is common to write $\lambda - A$, $(\lambda - A)^{-1}$ instead of $\lambda I - A$, $(\lambda I - A)^{-1}$. For $\lambda \in \rho(A)$ we have

$$(\lambda - A)R(\lambda, A) = I_X, \quad R(\lambda, A)(\lambda - A) = I_{D(A)}.$$

Further Remarks: (a) If $\rho(A) \neq \emptyset$ then A is closed.

(b) If A is closed then $\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A : D(A) \to X \text{ is bijective}\}$.

Proof. (a) We find $\lambda_0 \in \rho(A)$. Then $-R(\lambda_0, A) = -(\lambda_0 - A)^{-1}$ is closed. By 8.4(c), $A - \lambda_0 I$ is closed, and by 8.4(a), $A = (A - \lambda_0 I) + \lambda_0 I$ is closed.

(b) "$\subseteq$" is clear. For "$\supseteq$" assume that $\lambda - A : D(A) \to X$ is bijective. By 8.4(a), $\lambda - A$ is closed, and by 8.4(c), $(\lambda - A)^{-1} : X \to X$ is closed. By the Closed Graph Theorem we obtain $(\lambda - A)^{-1} \in \mathcal{L}(X)$, i.e. $\lambda \in \rho(A)$. □

Examples: 1) $X = C[0,1]$, $A = \frac{d}{dx}$, $D(A) = C^1[0,1]$. Let $\lambda \in \mathbb{C}$. Then $f := e^{\lambda(\cdot)} \in D(A)$ and $Af = f' = \lambda e^{\lambda(\cdot)} = \lambda f$, hence $\lambda - A : D(A) \to X$ is not injective. We have shown $\sigma(A) = \mathbb{C}$.

2) $X = C[0,1]$, $A_0 = \frac{d}{dx}$, $D(A_0) = \{f \in C^1[0,1] : f(0) = 0\}$. Let $\lambda \in \mathbb{C}$, $g \in C[0,1]$. By ODE-Theory there is a unique solution $f \in C^1[0,1]$ to the initial value problem

$$\lambda f - f' = g \text{ in } [0,1]; \quad f(0) = 0,$$

given by $f(t) := -\int_0^t e^{\lambda(t-s)} g(s)\, ds$, $t \in [0,1]$. Thus $\lambda - A_0 : D(A_0) \to X$ is bijective. Since A_0 is closed (!), we have shown $\sigma(A_0) = \emptyset$, $\rho(A) = \mathbb{C}$. Moreover, we have

$$|f(t)| \leq \int_0^t e^{Re\,\lambda(t-s)} |g(s)|\, ds \leq \int_0^1 e^{Re\,\lambda s}\, ds\, \|g\|_\infty = \begin{cases} \frac{e^{Re\,\lambda}-1}{Re\,\lambda} \|g\|_\infty & , Re\,\lambda \neq 0 \\ \|g\|_\infty & , Re\,\lambda = 0 \end{cases},$$

which means

$$\|R(\lambda, A_0)\| \leq \begin{cases} \frac{e^{Re\,\lambda}-1}{Re\,\lambda} & , Re\,\lambda \neq 0 \\ 1 & , Re\,\lambda = 0 \end{cases}.$$

Notice that this implies $\sigma(A_0) = \emptyset$ without using closedness of A_0.

8.6. Proposition (Properties of the resolvent): Let A be a closed linear operator in X. Then:

(a) For all $\lambda, \mu \in \rho(A)$:

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A) \quad (\textit{resolvent equation}).$$

In particular, $R(\lambda, A)R(\mu, A) = R(\mu, A)R(\lambda, A)$ for all $\lambda, \mu \in \rho(A)$.

(b) If $\lambda \in \rho(A)$ then $B(\lambda, 1/\|R(\lambda, A)\|) \subseteq \rho(A)$ and

$$R(\mu, A) = \sum_{k=0}^{\infty} (-1)^k R(\lambda, A)^{k+1} (\mu - \lambda)^k, \quad \mu \in B(\lambda, 1/\|R(\lambda, A)\|),$$

where the series converges in operator norm. In particular, $\rho(A)$ is open and $\sigma(A)$ is closed, and, for any $\lambda \in \rho(A)$,

$$\|R(\lambda, A)\| \geq \frac{1}{d(\lambda, \sigma(A))}.$$

Proof. (a) Write

$$R(\lambda,A)-R(\mu,A) = R(\lambda,A)\underbrace{(\mu-A)R(\mu,A)}_{=I_X} - \underbrace{R(\lambda,A)(\lambda-A)}_{=I_{D(A)}}R(\mu,A) = (\mu-\lambda)R(\lambda,A)R(\mu,A).$$

(b) For $\mu \in \mathbb{C}$ we have

$$\mu - A = \mu - \lambda + \lambda - A = ((\mu-\lambda)R(\lambda,A) + I)(\lambda - A).$$

For $|\mu - \lambda| < 1/\|R(\lambda,A)\|$ the operator

$$I + (\mu-\lambda)R(\lambda,A) \in \mathcal{L}(X)$$

is invertible in $\mathcal{L}(X)$ by a Neumann series and

$$(I + (\mu-\lambda)R(\lambda,A))^{-1} = \sum_{k=0}^{\infty}(-1)^k R(\lambda,A)^k(\mu-\lambda)^k.$$

We obtain for these μ that $\mu - A : D(A) \to X$ is bijective and

$$(\mu-A)^{-1} = R(\lambda,A)(I+(\mu-\lambda)R(\lambda,A))^{-1} = \sum_{k=0}^{\infty}(-1)^k R(\lambda,A)^{k+1}(\mu-\lambda)^k$$

as claimed. □

— — **Detour: Analyticity** — —

Definition: Let $\Omega \subseteq \mathbb{C}$ be open and X a complex Banach space. A function $f : \Omega \to X$ is called *analytic* (or *holomorphic*) in Ω if, for every $z_0 \in \Omega$ the limit

$$f'(z_0) := \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

exists. In this case, the function $f' : \Omega \to X$ is called the (complex) *derivative* of f.

Remark: A holomorphic function is continuous.

Power series: A (formal) *power series* in X has the form

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

where $(a_k)_{k\in\mathbb{N}_0}$ is a sequence in X and $z_0 \in \mathbb{C}$.

Radius of convergence: Define $R \in [0,\infty]$ by $\frac{1}{R} := \limsup_{k\to\infty} \|a_k\|_X^{1/k}$. Then the power series above converges absolutely and uniformly on compact subsets of $B(z_0,R)$ and defines a holomorphic function f in $B(z_0,R)$ which satisfies

$$f^{(k)}(z_0) = k!a_k, \quad k \in \mathbb{N}_0.$$

Moreover, for $\tilde{R} > R$, f cannot be extended to a holomorphic function on $B(z_0,\tilde{R})$.

Proof. Let $\varepsilon \in (0, R/2)$ and choose $k_0 \in \mathbb{N}$ such that

$$\|a_k\|^{1/k} \le \frac{1}{R-\varepsilon} \text{ for all } k \ge k_0.$$

For $|z - z_0| \le R - 2\varepsilon$ we then have

$$\sum_{k=k_0}^{\infty} \|a_k\| |z - z_0|^k \le \sum_{k=k_0}^{\infty} \Big(\frac{R-2\varepsilon}{R-\varepsilon}\Big)^k < \infty.$$

Differentiability follows as for scalar-valued power series.

Now let $\tilde{R} \ge R$ such that f has an analytic extension $g : B(z_0, \tilde{R}) \to X$. For arbitrary $\varphi \in X'$, the function $\varphi \circ g : B(z_0, \tilde{R}) \to \mathbb{C}$ is analytic and

$$\sum_{k=0}^{\infty} \varphi(a_k)(z - z_0)^k$$

converges for all $|z - z_0| < \tilde{R}$. Hence for any $\delta \in (0, \tilde{R})$,

$$\sup_k |\varphi(a_k)|(\tilde{R} - \delta)^k < \infty.$$

By the Uniform Boundedness Principle we conclude that

$$C_\delta := \sup_k \|a_k\|(\tilde{R} - \delta)^k < \infty \text{ for any } \delta \in (0, \tilde{R}),$$

which implies

$$\|a_k\|^{1/k} \le \frac{C_\delta^{1/k}}{\tilde{R} - \delta} \text{ for all } \delta \in (0, \tilde{R}),\ k \in \mathbb{N},$$

and

$$\frac{1}{R} = \limsup_k \|a_k\|^{1/k} \le \frac{1}{\tilde{R} - \delta} \text{ for all } \delta \in (0, \tilde{R}).$$

Letting $\delta \to 0$, we obtain $\tilde{R} \le R$. □

end Thu 25.04.19

Remark: The proof indicates the two basic ways of approaching holomorphic functions with values in a Banach space. The first is to proceed as in the complex valued case, and the second is to apply linear functionals and resort to results on complex valued functions.

The following will be an exercise:

Proposition: A function $f : \Omega \to X$ is analytic if and only if it is *weakly analytic*, i.e. $\varphi \circ f : \Omega \to \mathbb{C}$ is analytic for every $\varphi \in X'$.

— — **End of the Detour** — —

For the applications we need another lemma.

8.7. Lemma: Let (c_n) be a real sequence such that $0 \le c_{n+m} \le c_n c_m$ for all $n, m \in \mathbb{N}$. Then $\sqrt[n]{c_n} \to c := \inf_k \sqrt[k]{c_k}$ as $n \to \infty$.

Proof. Let $\varepsilon > 0$ and choose m such that $\sqrt[m]{c_m} < c + \varepsilon$. Let $b := \max\{c_1, \dots, c_m\}$ and write $n > m$ as $n = km + r$ with $k \in \mathbb{N}$ and $r \in \{1, \dots, m\}$. Then

$$c_n^{1/n} = (c_{km+r})^{1/n} \le (c_m^k c_r)^{1/n} \le (c+\varepsilon)^{km/n} b^{1/n} = (c+\varepsilon)(c+\varepsilon)^{-r/n} b^{1/n} \le c + 2\varepsilon$$

for large n, since $(c+\varepsilon)^{-r/n} b^{1/n} \to 1$ as $n \to \infty$ (here we assumed $b > 0$ without loss of generality). □

8.8. Corollary: If A is closed linear operator in X, then its resolvent $\rho(A) \to \mathcal{L}(X)$, $\lambda \mapsto R(\lambda, A)$, is an analytic function and

$$\frac{d^k}{d\lambda^k} R(\lambda, A) = (-1)^k k! R(\lambda, A)^{k+1}, \quad k \in \mathbb{N}_0.$$

For $\lambda \in \rho(A)$ we have

$$d(\lambda, \sigma(A)) = \frac{1}{\inf_k \|R(\lambda, A)^k\|^{1/k}}. \tag{+}$$

Proof. The first part is clear from 8.6(b) and analyticity of power series. Applying 8.7 to $c_k := \|R(\lambda, A)^k\|$ we see that the right hand side of (+) is the radius of convergence R for the power series in 8.6(b). By the above, $\partial B(\lambda, R) \cap \sigma(A) \ne \emptyset$ (otherwise the resolvent would be holomorphic on a strictly larger ball which is impossible). □

We give an application to bounded operators.

8.9. Proposition and Definition: Let $T \in \mathcal{L}(X)$ and $X \ne \{0\}$. Then $\sigma(T) \ne \emptyset$, and the *spectral radius of* T,

$$r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}$$

satisfies

$$r(T) = \lim_{k \to \infty} \|T^k\|^{1/k} = \inf_k \|T^k\|^{1/k}.$$

In particular, $\sigma(T)$ is a non-empty, compact subset of $B(0, \|T\|)$.

Proof. We begin with the observation that, for $|\lambda| > \|T\|$,

$$\lambda I - T = \lambda\Big(I - \frac{T}{\lambda}\Big),$$

which is invertible by a Neumann series

$$(\lambda I - T)^{-1} = \lambda^{-1} \sum_{k=0}^{\infty} \frac{T^k}{\lambda^k} = \sum_{k=0}^{\infty} T^k \lambda^{-(k+1)}.$$

The latter is a power series in λ^{-1} with radius of convergence $R := 1/\inf_k \|T^k\|^{1/k}$. Hence it converges for $|\lambda| > 1/R$, and $\sigma(T) \subset \overline{B}(0, 1/R)$, i.e. $r(T) \leq 1/R$. The above on power series also shows that $r(T) = 1/R$ if $1/R > 0$.

We now show that $\sigma(T) \neq \emptyset$ (then $\sigma(T) = \{0\}$ in case $1/R = 0$). From the series representation above we also obtain that $R(\lambda, T) \to 0$ as $|\lambda| \to \infty$. If $\sigma(T) = \emptyset$ then, for any $\varphi \in \mathcal{L}(X)'$, $\lambda \to \varphi \circ R(\lambda, T)$ is an entire function, which is bounded (since it tends to 0 as $|\lambda| \to \infty$). By Liouville's Theorem, this function is constant. We conclude $\varphi \circ R(\lambda, T) = 0$ for any λ and any φ. But then $R(\lambda, T) = 0$ which implies $X = \{0\}$, and this case was excluded. □

Bemerkung: There exist operators T with $\|T\| = 1$, $r(T) = 0$, and hence $\sigma(T) = \{0\}$, e.g. $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in $X = \mathbb{C}^2$ (equipped with Euclidean norm).

8.10. Definition (Fine structure of the spectrum): Let A be a closed linear operator in X. We define

$\sigma_p(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not injective}\} = \{\lambda \in \mathbb{C} : N(\lambda I - A) \neq \{0\}\}$ (*point spectrum*).

Any $\lambda \in \sigma_p(A)$ is called an *eigenvalue of* A and any $x \in N(\lambda I - A) \setminus \{0\}$ is called an *eigenvector* for the eigenvalue λ. We further define

$$\begin{aligned} \sigma_r(A) &:= \{\lambda \in \mathbb{C} : R(\lambda I - A) \text{ is not dense in } X\} \text{ (residual spectrum)} \\ \sigma_c(A) &:= \{\lambda \in \mathbb{C} : R(\lambda I - A) \text{ is dense but not closed in } X\} \text{ (continuous spectrum)} \\ \sigma_{ap}(A) &:= \{\lambda \in \mathbb{C} : \text{ there exists a sequence } (x_n) \text{ in } D(A) \text{ with } \|x_n\| = 1 \text{ for all } n \\ &\qquad \text{and } (\lambda I - A)x_n \to 0\} \text{ (approximate point spectrum)}. \end{aligned}$$

A sequence (x_n) as in the definition of the approximate point spectrum is sometimes called an *approximate eigenvector*.

Remark: $\sigma_p(A) \subseteq \sigma_{ap}(A)$ (take $x_n = x$ eigenvector).

Examples: 1) If $\dim X < \infty$ and $A \in \mathcal{L}(X)$ then $\sigma(A) = \sigma_p(A) = \sigma_r(A)$ (A is a finite-dimensional matrix for which injectivity is equivalent to surjectivity).

2) Let $X = C[0, 1]$, $A = \frac{d}{dx}$ with $D(A) = C^1[0, 1]$. We saw above that $\sigma(A) = \sigma_p(A) = \mathbb{C}$. On the other hand, for any $\lambda \in \mathbb{C}$ the ordinary differential equation $\lambda f - f' = g$ has a (non-unique) solution $f \in C^1[0, 1]$ for each right hand side $g \in C[0, 1]$. Hence $R(\lambda I - A) = X$ for each $\lambda \in \mathbb{C}$ and $\sigma_r(A) = \sigma_c(A) = \emptyset$.

8.11. Proposition: Let A be a closed operator in X. Then:

(a) $\sigma_{ap}(A) = \sigma_p(A) \cup \{\lambda \in \mathbb{C} : \lambda I - A \text{ is injective and } (\lambda I - A)^{-1} \text{ is not bounded}\}$.

(b) $\sigma(A) = \sigma_{ap}(A) \cup \sigma_r(A)$.

(c) $\partial\sigma(A) \subseteq \sigma_{ap}(A)$.

Proof. (a) Let $\lambda \in \mathbb{C}$ and let $\lambda - A$ be injective, $Y := R(\lambda - A)$. Then:

$$\begin{aligned} & (\lambda - A)^{-1} : Y \to X \text{ is not bounded} \\ \Leftrightarrow\ & \text{there exists } (y_n) \text{ in } Y \text{ such that } \|y_n\| = 1 \text{ and } \|(\lambda - A)^{-1} y_n\| \to \infty \\ \Leftrightarrow\ & \text{there exists } (z_n) \text{ in } Y \text{ such that } z_n \to 0 \text{ and } \|(\lambda - A)^{-1} z_n\| = 1 \\ \Leftrightarrow\ & \text{there exists } (x_n) \text{ in } D(A) \text{ s.t. } \|x_n\| = 1 \text{ and } (\lambda - A)x_n \to 0 \end{aligned}$$

To see this put $z_n = y_n/\|(\lambda - A)^{-1} y_n\|$, $y_n = z_n/\|z_n\|$, and $x_n = (\lambda - A)^{-1} z_n$, $z_n = (\lambda - A)x_n$, respectively. Thus we have shown

$$\sigma_{ap}(A) \setminus \sigma_p(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is injective and } (\lambda I - A)^{-1} \text{ is not bounded}\}.$$

(b) "$\supseteq$" is clear. For the proof of "$\subseteq$" let $\lambda \in \sigma(A) \setminus \sigma_{ap}(A)$. By (a), $\lambda - A$ is injective and $(\lambda - A)^{-1}$ is bounded. In particular, $R(\lambda - A)$ is closed in X. Hence $\overline{R(\lambda - A)} = R(\lambda - A) \neq X$ (otherwise $\lambda \notin \sigma(A)$!). We have shown $\lambda \in \sigma_r(A)$.

(c) Let $\lambda \in \partial\sigma(A)$. We find a sequence (λ_n) in $\rho(A)$ with $\lambda_n \to \lambda$. By 8.6(b) we have $\|R(\lambda_n, A)\| \to \infty$. By the Uniform Boundedness Principle we find $y \in X$ such that $\alpha_n := \|R(\lambda_n, A)y\| \to \infty$ (observe that we have $\alpha_n > 0$ for any n). Let $x_n := R(\lambda_n, A)y/\alpha_n$, $n \in \mathbb{N}$. Then (x_n) is a sequence in $D(A)$ with $\|x_n\| = 1$ for all n. Moreover,

$$(\lambda - A)x_n = (\lambda - \lambda_n)x_n + (\lambda_n - A)x_n = (\lambda - \lambda_n)x_n + y/\alpha_n \to 0 \quad (n \to \infty).$$

□

Example: Let $X = l^2 = \{(x_n)_{n\in\mathbb{N}} : \|(x_n)\|l^2 := \left(\sum_n |x_n|^2\right)^{1/2} < \infty\}$ and let $L \in \mathcal{L}(l^2)$ be the *left shift* given by $L(x_n) := (x_{n+1})$, i.e. $L(x_n) = (x_2, x_3, x_4, \ldots)$. Clearly, we have $\|L\| = 1$, $\|L^k\| = 1$ for every $k \in \mathbb{N}$, and hence $r(L) = 1$. For $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ we have

$$(\lambda - L)(x_n) = 0 \Leftrightarrow \forall n : \lambda x_n - x_{n+1} = 0 \Leftrightarrow \forall n : x_{n+1} = \lambda^n x_1.$$

For $|\lambda| < 1$ we have $(1, \lambda, \lambda^2, \ldots) \in l^2$, hence $\lambda \in \sigma_p(L)$. For $|\lambda| = 1$ we have $(1, \lambda, \lambda^2, \ldots) \notin l^2$, and $\lambda \notin \sigma_p(L)$.

We conclude $\sigma_p(L) = \{|\lambda| < 1\}$, $\sigma(L) = \sigma_{ap}(L) = \{|\lambda| \leq 1\}$, and $\sigma_{ap}(L) \setminus \sigma_p(L) = \{|\lambda| = 1\} = \partial\sigma(L)$. end Tue 30.04.19

8.12. Dual operators: Recall that, for $T \in \mathcal{L}(X, Y)$, its *dual* (or *adjoint*)[2] *operator* $T' \in \mathcal{L}(X', Y')$ is given by

$$T'\phi := \phi \circ T, \quad \phi \in Y'.$$

[2]In this course we will reserve the term "adjoint" for the *Hilbert space adjoint* and thus speak of "dual operators" here.

Using *duality brackets*

$$\langle y, \phi\rangle_{Y\times Y'} := \phi(y), \quad \text{for all } y \in Y,\ \phi \in Y',$$

this can be written as

$$\langle x, T'\phi\rangle_{X\times X'} = \langle Tx, \phi\rangle_{Y\times Y'} \quad \text{for all } x \in X,\ \phi \in Y',$$

Example: Let $X = Y = l^2$ and L be the left shift. We calculate L'. For $X = l^2$ we have $X' = l^2$ where the duality bracket is given by

$$\langle (x_n), (y_n)\rangle_{l^2\times l^2} = \sum_n x_n y_n.$$

For $(x_n), (y_n) \in l^2$ we then have

$$\langle L(x_n), (y_n)\rangle = \sum_{n=1}^{\infty} x_{n+1} y_n = \sum_{n=2}^{\infty} x_n y_{n-1} = \langle (x_n), (y_{n-1})\rangle$$

if we let $y_0 := 0$. Hence $L' = R$ where R is the *right shift* given by

$$R(y_n) := (0, y_1, y_2, y_3, \ldots).$$

Clearly, we have $\|R(y_n)\|_{l^2} = \|(y_n)\|_{l^2}$, $\|R\| = 1$, $r(T) = 1$, and $\sigma(R) \subseteq \{|\lambda| \leq 1\}$. For $|\lambda| \leq 1$ we have

$$(\lambda - R)(y_n) = 0 \Leftrightarrow \lambda y_1 = 0, \forall n \geq 2 : \lambda y_n = y_{n-1} \Leftrightarrow (y_n) = 0.$$

Hence $\sigma_p(R) = \emptyset$.

Rules for dual operators: The following rules are easily verified:

$$(I_X)' = I_{X'}, \quad (S+T)' = S' + T', \quad (\alpha T)' = \alpha T', \quad (ST)' = T'S'$$

where $T \in \mathcal{L}(X, Y)$, $\alpha \in \mathbb{C}$, $S \in \mathcal{L}(X, Y)$ for the sum and $S \in \mathcal{L}(Y, Z)$ for the product, respectively.

Remark: For $T \in \mathcal{L}(X, Y)$ we have

$$T : X \to Y \text{ is an isomorphism} \Longleftrightarrow T' : Y' \to X' \text{ is an isomorphism.}$$

Moreover, we have $(T')^{-1} = (T^{-1})'$ in this case.

Proof. "$\Rightarrow$": We have

$$T'(T^{-1})' = (T^{-1}T)' = (I_X)' = I_{X'}, \quad (T^{-1})'T' = (TT^{-1})' = (I_Y)' = I_{Y'},$$

which proves invertibility of T' and $(T')^{-1} = (T^{-1})'$.

"$\Leftarrow$": Suppose that T' has an inverse $S : X' \to Y'$. Then $T'' := (T')'$ is an isomorphism $X'' \to Y''$ by the first part of the proof. Since X is a closed subspace of X'', $T''(X)$ is a closed subspace of Y''. But $T''(X) = T(X) \subseteq Y$, so $T(X)$ is a closed subspace of Y.

If $\phi \in X'$ satisfies $\phi|_{T(X)} = 0$ then

$$\langle x, T'\phi\rangle = \langle Tx, \phi\rangle = 0, \quad x \in X,$$

i.e. $T'\phi = 0$. Since T' is injective we conclude $\phi = 0$. By Hahn-Banach we thus have shown that $T(X)$ is dense in Y. Since $T(X)$ is also closed in Y we obtain $T(X) = Y$, i.e. T is surjective.

Since T'' is injective, $T = T''|_X$ is injective. We have shown that $T : X \to Y$ is bijective. By the Open Mapping Theorem, $T : X \to Y$ is an isomorphism. $\square$

Recall for the previous proof: The *bidual* $X'' = (X')' = \mathcal{L}(X', \mathbb{C})$ of X is a Banach space, and the map

$$J : X \to X'',\ x \mapsto \delta_x \quad \text{where } \delta_x : X' \to \mathbb{C}, \phi \mapsto \delta_x(\phi) := \phi(x)$$

is (by Hahn-Banach) an isometric injection. Written with duality brackets we have

$$\langle \phi, \delta_x\rangle_{X' \times X''} = \langle x, \phi\rangle_{X \times X'}, \quad x \in X, \phi \in X'.$$

It is common to identify X with the closed subspace $J(X)$ of X''. The space X is called *reflexive* if $J(X) = X''$.

If $T \in \mathcal{L}(X, Y)$ then $T'' := (T')' \in \mathcal{L}(X'', Y'')$ and $T = T''|_X$, since for $x \in X$, $\phi \in Y'$ we have

$$\langle \phi, T''\delta_x\rangle = \langle T'\phi, \delta_x\rangle = \langle x, T'\phi\rangle = \langle Tx, \phi\rangle = \langle \phi, \delta_{Tx}\rangle,$$

which means $T''\delta_x = \delta_{Tx}$ for all $x \in X$, i.e. $T''x = Tx$, $x \in X$, if we identify X and $J(X)$.

8.13. Definition: Let A be a linear operator from X to Y with $D(A)$ dense in X. We define the linear operator $A' : Y' \supseteq D(A') \to X'$ by letting for $\phi \in Y'$, $\psi \in X'$:

$$\phi \in D(A'), A'\phi = \psi :\Longleftrightarrow \forall x \in D(A) : \langle Ax, \phi\rangle_{Y \times Y'} = \langle x, \psi\rangle_{X \times X'}$$

Then $D(A')$ is the set of all $\phi \in Y'$ such that the map

$$D(A) \to \mathbb{C},\ x \mapsto \langle Ax, \phi\rangle,$$

has a continuous extension $\psi \in X'$. By denseness of $D(A)$ in X, this extension is uniquely determined (if it exists). Then $A'\phi$ is this unique extension ψ.

Remark: A' is a **always** closed: Let (ϕ_n) be a sequence in $D(A')$ such that $\phi_n \to \phi$ in Y' and $A'\phi_n \to \psi$ in X'. Then we have, for any $x \in D(A)$:

$$\langle Ax, y\rangle = \lim_n \langle Ax, \phi_n\rangle = \lim_n \langle x, A'\phi_n\rangle = \langle x, \psi\rangle,$$

which means $\phi \in D(A')$ and $A'\phi = \psi$.

Example: $X = L^2(0,1)$, $A = \frac{d}{dx}$, $D(A) = C_0^1[0,1] := \{\varphi \in C^1[0,1] : \varphi(0) = \varphi(1) = 0\}$. Then

$$D(A') = \{f \in L^2(0,1) : \exists g \in L^2(0,1) \forall \varphi \in C_0^1[0,1] : \int_0^1 f\varphi'\,dx = \int_0^1 g\varphi\,dx\}.$$

In particular we see that any $f \in D(A')$ has a weak derivative in $L^2(0,1)$ and that $A'f = -f'$, $D(A') \subseteq W^{1,2}(0,1)$. On the other hand, if $f \in W^{1,2}(0,1)$ then we have

$$\int_0^1 f\varphi'\,dx = -\int_0^1 f'\varphi\,dx$$

for all $\varphi \in C_c^\infty(0,1)$. Any $\varphi \in C_0^1[0,1]$ can be approximated by a sequence (φ_n) in $C_c^\infty(0,1)$ in $W^{1,2}$-norm. This yields $D(A') = W^{1,2}(0,1)$, $A' = -\frac{d}{dx}$ (weak derivative).

Rules: $(\lambda - A)' = \lambda - A'$ for $\lambda \in \mathbb{C}$ and $X = Y$ (more general one has $(A+T)' = A' + T'$ if $T \in \mathcal{L}(X,Y)$). If B is an extension of A then A' is an extension of B': $A \subseteq B \Rightarrow B' \subseteq A'$.

8.14. Proposition: Let A be a closed linear operator in X which is densely defined. Then $\sigma(A') = \sigma(A)$ and $\sigma_p(A') = \sigma_r(A)$.

Proof. We start with the second assertion. For $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} &\lambda \in \sigma_r(A) \;\Leftrightarrow\; \overline{(\lambda - A)X} \neq y \;\Leftrightarrow\; \exists \phi \in Y' \setminus \{0\} \forall x \in D(A) : \langle (\lambda - A)x, \phi\rangle = 0 \\ \Leftrightarrow\;& \exists \phi \in D(A') \setminus \{0\} : (\lambda - A')\phi = 0 \;\Leftrightarrow\; \lambda \in \sigma_p(A'). \end{aligned}$$

$\sigma(A') \subseteq \sigma(A)$: Let $\lambda \in \rho(A)$. By what we have just proved, $\lambda \notin \sigma_r(A) = \sigma_p(A')$ and $\lambda - A'$ is injective. Now let $\psi \in X'$ and set $\phi := R(\lambda, A)'\psi$. For $x \in D(A)$ we then have

$$\langle (\lambda - A)x, \phi\rangle = \langle R(\lambda, A)(\lambda - A)x, \psi\rangle = \langle x, \psi\rangle.$$

This means $\phi \in D((\lambda - A)') = D(A')$ and $(\lambda - A')\phi = \psi$. Hence $\lambda - A'$ is also surjective, and $\lambda \in \rho(A)$.

$\sigma(A) \subseteq \sigma(A')$: Let $\lambda \in \rho(A')$. Then $\lambda \notin \sigma_p(A') = \sigma_r(A)$, and thus $Y := (\lambda - A)(D(A))$ is dense in X.

We let $S := R(\lambda, A') \in \mathcal{L}(X')$. Then $S' \in \mathcal{L}(X'')$ and we define $R := S'|_X$. We claim that $R \in \mathcal{L}(X)$ is the inverse of $\lambda - A$. For $x \in D(A)$ and $\phi \in X'$ we have

$$\langle \phi, S'\delta_{(\lambda-A)x}\rangle = \langle S\phi, \delta_{(\lambda-A)x}\rangle = \langle (\lambda - A)x, S\phi\rangle.$$

Since $S\phi \in D(A')$ we can continue

$$\langle (\lambda - A)x, S\phi\rangle = \langle x, (\lambda - A')S\phi\rangle = \langle x, \phi\rangle = \langle \phi, \delta_x\rangle.$$

We thus have shown $S'\delta_{(\lambda-A)x} = \delta_x$ for all $x \in D(A)$. In particular R maps the dense range of $\lambda - A$ into X, hence $R \in \mathcal{L}(X)$. Moreover, $R(\lambda - A)x = x$ for all $x \in D(A)$. Hence $\lambda - A$ is injective and $Ry = (\lambda - A)^{-1}y$ for all $y \in Y$. Thus we see that $(\lambda - A)^{-1}$ is a bounded operator. Since it is also closed, its domain Y has to be a closed subspace of X. By denseness of Y we thus obtain $Y = X$, $\lambda \in \rho(A)$ and $R = R(\lambda, A)$. □

Example: Let $X = l^2$, L be the left shift, and R be the right shift. Since $L' = R$, we obtain $\sigma_r(L) = \sigma_p(R) = \emptyset$.

The spectrum of an operator is related to the spectrum of its resolvents. This is called *spectral mapping*.

8.15. Proposition: Let A be a closed linear operator in X and $\lambda_0 \in \rho(A)$. Then

$$\sigma(R(\lambda_0, A)) \setminus \{0\} = \left\{ \frac{1}{\lambda_0 - z} : z \in \sigma(A) \right\}$$

and $0 \in \sigma(R(\lambda_0, A))$ if and only if $A \notin \mathcal{L}(X)$.

Proof. We have

$$0 \notin \sigma(R(\lambda_0, A)) \;\Leftrightarrow\; R(\lambda_0, A) : X \to X \text{ surjective} \;\Leftrightarrow\; D(A) = X \;\Leftrightarrow\; A \in \mathcal{L}(X),$$

where we used the Closed Graph Theorem for the last equivalence. Now let $z \neq \lambda_0$. Then

$$z - A = z - \lambda_0 + \lambda_0 - A = ((z - \lambda_0)R(\lambda_0, A) + I)(\lambda_0 - A) = \Big(R(\lambda_0, A) - \frac{1}{\lambda_0 - z}\Big)(z - \lambda_0)(\lambda_0 - A),$$

where $(z - \lambda_0)(\lambda_0 - A) : D(A) \to X$ is bijective. Hence $z - A : D(A) \to X$ is bijective if and only if $R(\lambda_0, A) - (\lambda_0 - z)^{-1} : X \to X$ is bijective. We obtain that $z \in \sigma(A)$ is equivalent to $(\lambda_0 - z)^{-1} \in \sigma(R(\lambda_0, A))$. □

Remark: From the proof we see that, for $\lambda_0, z \in \rho(A)$ with $z \neq \lambda_0$:

$$\Big(\frac{1}{\lambda_0 - z} - R(\lambda_0, A)\Big)^{-1} = (\lambda_0 - z)(\lambda_0 - A)R(z, A).$$

What can be done if a linear operator A from X to Y is not closed? Of course, one can consider the closure $\overline{\mathrm{gr}(A)}$ in $X \times Y$. But this need not be the graph of an operator.

8.16. Definition: A linear operator A from X to Y is called *closable* if it has a closed linear extension $B : X \supseteq D(B) \to Y$. In this case, A has a smallest closed linear extension $\overline{A}$ which is called the *closure* of A.

The following properties are equivalent (this will be an exercise):

- A is closable,
- $\overline{\mathrm{gr}(A)}$ is the graph of a function,
- if (x_n) is a sequence in $D(A)$ such that $x_n \to 0$ in X and $Ax_n \to y$ in Y then $y = 0$.

In this case, one has $\mathrm{gr}(\overline{A}) = \overline{\mathrm{gr}(A)}$. end Thu 02.05.19

8.17. Proposition: Let A be a densely defined linear operator in X. If $D(A')$ is dense in X' then A is closable. The converse holds if X is reflexive.

Proof. If $D(A')$ is dense in X' then $A'' := (A')'$ is a closed linear operator in X''. We show that A'' is an extension of A: For $x \in D(A)$, $\phi \in D(A')$ we have

$$\langle Ax, \phi \rangle = \langle x, A'\phi \rangle = \langle A'\phi, \delta_x \rangle,$$

which shows that $\delta_x \in D(A'')$ and $A''\delta_x = \delta_{Ax}$ for all $x \in D(A)$.

Now let X be reflexive and assume that $D(A')$ is not dense in X'. By Hahn-Banach we find $y \in X'' = X$, $y \neq 0$, such that

$$\langle y, \phi \rangle = 0 = \langle 0, A'\phi \rangle \quad \text{for all } \phi \in D(A').$$

We claim that $(0, y) \in \overline{\mathrm{gr}(A)}$. If $(\psi, \phi) \in X' \times X'$ such that $(\psi, \phi)|_{\mathrm{gr}(A)} = 0$. Then $\phi \in D(A')$ and $A'\phi = -\psi$. From the property of y above we get $(\psi, \phi)(0, y) = 0$. By Hahn-Banach we conclude that $(0, y) \in \overline{\mathrm{gr}(A)}$ as claimed. By $y \neq 0$, A is not closable. □

9 Spectral projections and holomorphic functional calculus

When nothing else is said, X denotes a complex Banach space and A is a closed linear operator in X.

9.1. Spectral projections: Suppose that $\sigma(A) = \sigma_0 \cup \sigma_1$ where $\sigma_0 \cap \sigma_1 = \emptyset$, σ_0 is compact, and σ_1 is closed. Then we find finitely many closed piecewise C^1-curves Γ in $\mathbb{C} \setminus \sigma(A)$ such that

$$n(z, \Gamma) = \begin{cases} 1 & , z \in \sigma_0 \\ 0 & , z \in \sigma_1 \end{cases} ,$$

where $n(z, \Gamma) := \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\zeta - z}$ denotes the *winding number*. Let

$$P := \frac{1}{2\pi i} \int_\Gamma R(\lambda, A)\, d\lambda = \frac{1}{2\pi i} \int_0^1 R(\gamma(t), A)\dot{\gamma}(t)\, dt$$

where $\gamma : [0, 1] \to \mathbb{C}$ is a parametrization of Γ and the integral is a Riemann integral (or a Bochner integral).

The operator P has the following properties:

(a) $P \in \mathcal{L}(X)$ and $P^2 = P$, i.e. P is a projection.

(b) P commutes with A, i.e. for $x \in D(A)$ we have $Px \in D(A)$ and $APx = PAx$. Moreover, $X_0 := R(P) \subseteq D(A)$.

(c) If we let $A_0 := A|_{X_0}$ then $A_0 \in \mathcal{L}(X_0)$ and $\sigma(A_0) = \sigma_0$.

(d) Let $X_1 := N(P)$ and define A_1 as the restriction of A to $D(A_1) := D(A) \cap X_1$. Then A_1 is a closed linear operator in X_1 and $\sigma(A_1) = \sigma_1$.

This means that P induces a decomposition of the space X into the subspaces X_0 and X_1 which are complements of each other and invariant under A.

For the proof we shall need the following.

9.2. Holomorphic functions with values in a Banach space: Cauchy's theorem and the Cauchy integral formula hold for X-valued holomorphic functions $f : \Omega \to X$:

$$\int_\Gamma f(z)\, dz = 0 \quad \text{and} \quad \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\, dz = f(z_0)$$

if Γ is a finite family of closed piecewise C^1-curves such that $n(z, \Gamma) = 0$ for all $z \notin \Omega$ and $n(z_0, \Gamma) = 1$.

Proof of 9.1. (a) $P \in \mathcal{L}(X)$ is well defined since $t \mapsto R(\gamma(t), A)\dot{\gamma}(t)$ is piecewise continuous with values in $\mathcal{L}(X)$. By Cauchy's theorem, Γ does not depend on the special choice of Γ, so we choose another family Γ' such that $n(\lambda, \Gamma') = 1$ for all $\lambda \in \Gamma$ (Γ' encircles all points of Γ once) and $n(\mu, \Gamma) = 0$ for all $\mu \in \Gamma'$. This means that Γ is "inside" Γ'. Then, by the resolvent equation and 9.2,

$$\begin{aligned} P^2 &= \Big(\frac{1}{2\pi i}\Big)^2 \int_\Gamma \int_{\Gamma'} R(\lambda, A) R(\mu, A)\, d\mu\, d\lambda \\ &= \Big(\frac{1}{2\pi i}\Big)^2 \int_\Gamma \int_{\Gamma'} \frac{R(\lambda, A) - R(\mu, A)}{\mu - \lambda}\, d\mu\, d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma \underbrace{\frac{1}{2\pi i} \int_{\Gamma'} \frac{1}{\mu - \lambda}\, d\mu}_{=1}\, R(\lambda, A)\, d\lambda + \Big(\frac{1}{2\pi i}\Big)^2 \int_{\Gamma'} \underbrace{\int_\Gamma \frac{R(\mu, A)}{\mu - \lambda}\, d\lambda}_{=0}\, d\mu \\ &= \frac{1}{2\pi i} \int_\Gamma R(\lambda, A)\, d\lambda = P. \end{aligned}$$

(b) Actually, $t \mapsto R(\gamma(t), A)\dot{\gamma}(t)$ is piecewise continuous with values in $\mathcal{L}(X, [D(A)])$, so $P \in \mathcal{L}(X, [D(A)])$, in particular $R(P) \subseteq D(A)$. We thus have

$$AP = \frac{1}{2\pi i} \int_\Gamma AR(\lambda, A)\, d\lambda,$$

where $t \mapsto AR(\gamma(t), A)\dot{\gamma}(t)$ is pieceweise continuous with values in $\mathcal{L}(X)$. For $x \in D(A)$ we have $AR(\lambda, A)x = R(\lambda, A)Ax$, which implies $APx = PAx$.

(c) and (d): Since $P \in \mathcal{L}(X)$ is a projection, $X_0 = R(P)$ is a closed subspace of X. By (b) we have $A_0 : X_0 \to X_0$. By closedness of A, this operator is closed, hence $A_0 \in \mathcal{L}(X_0)$. Clearly, P commutes with resolvents of A, which implies that X_0 is invariant under resolvents of A. By an exerxcise we thus have $\sigma(A_0) \subseteq \sigma(A)$, and $R(\lambda, A_0) = R(\lambda, A)|_{X_0}$ for $\lambda \in \rho(A)$.

On the other hand, for $\mu \in \sigma_1$ let

$$R_\mu := \frac{1}{2\pi i} \int_\Gamma \frac{R(\lambda, A)}{\mu - \lambda}\, d\lambda.$$

Then $R_\mu|_{X_0} \in \mathcal{L}(X_0)$, and by $AR(\lambda, A) = \lambda R(\lambda, A) - I_X$ we have

$$(\mu - A_0) R_\mu|_{X_0} = \frac{1}{2\pi i} \int_\Gamma \frac{(\mu - A) R(\lambda, A)|_{X_0}}{\mu - \lambda}\, d\lambda = P|_{X_0} - \frac{1}{2\pi i} \int_\Gamma \frac{I_{X_0}}{\mu - \lambda}\, d\lambda = I_{X_0}.$$

Clearly, $R_\mu|_{X_0}$ commutes with $\mu - A_0$, so also $R_\mu|_{X_0}(\mu - A_0) = I_{X_0}$. Hence $\mu \in \rho(A_0)$ and $R(\mu, A_0) = R_\mu|_{X_0}$. We thus have shown $\sigma(A_0) \subseteq \sigma_0$.

Similarly, X_1 is invariant under resolvents of A, and thus $\sigma(A_0) \subseteq \sigma(A)$. For $\mu \in \sigma_0$ let

$$R_\mu := \frac{1}{2\pi i} \int_\Gamma \frac{R(\lambda, A)}{\mu - \lambda}\, d\lambda.$$

Then $R_\mu \in \mathcal{L}(X,[D(A)])$ commutes with resolvents of A, hence with P, and $R_\mu|_{X_1} \in \mathcal{L}(X_1)$. For $x \in X_1$ we have

$$(\mu - A)R_\mu x = \underbrace{\frac{1}{2\pi i}\int_\Gamma R(\lambda,A)x\,d\lambda}_{=Px=0} + \frac{1}{2\pi i}\int_\Gamma \frac{x}{\mu-\lambda}\,d\lambda = -x.$$

Since R_μ commutes with A we obtain $\mu \in \rho(A_1)$ and $R(\mu,A_1) = -R_\mu|_{X_1}$. This means that we have shown $\sigma(A_1) \subseteq \sigma_1$.

The decomposition $X = X_0 \oplus X_1$ leads to the decomposition of A as $\begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}$ with domain $X_0 \oplus D(A_1)$. It is thus clear that $\sigma(A) = \sigma(A_0) \cup \sigma(A_1)$ (this union is closed since $\sigma(A_0)$ is compact). Since $\sigma(A_j) \subseteq \sigma_j$ ($j = 0,1$) and $\sigma_0 \cap \sigma_1 = \emptyset$ this is only possible if $\sigma(A_j) = \sigma_j$ for $j = 0,1$. $\square$

Example: Let $X = \mathbb{C}^n$ and $A \in \mathbb{C}^{n\times n}$, and let $\lambda_0 \in \sigma_p(A_0)$, $\sigma_0 = \{\lambda_0\}$, $\sigma_1 = \sigma(A) \setminus \{\lambda_0\}$. Then X_0 equals the *generalized eigenspace* $\bigcup_{k=1}^n N\big((\lambda_0 - A)^k\big)$ and

$$X_1 = \bigoplus_{\lambda\in\sigma_1} \bigcup_{k=1}^n N\big((\lambda - A)^k\big).$$

end Tue 07.05.19

9.3. The Dunford functional calculus: Let A be bounded in X. Then $\sigma(A)$ is compact.

(1) Let $U \subseteq \mathbb{C}$ be an open neighborhood of $\sigma(A)$. Then we find a finite family of closed piecewise C^1-curves Γ such that $n(z,\Gamma) = 1$ if $z \in \sigma(A)$ and $n(z,\Gamma) = 0$ for $z \notin U$. Let $f : U \to \mathbb{C}$ be a holomorphic function. Define

$$f(A) := \frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda,A)\,d\lambda.$$

Then $f(A) \in \mathcal{L}(X)$ and the definition does not depend on the special choice of Γ.

(2) The map $\Psi : f \mapsto f(A)$ is an algebra homomorphism, and

$$p_n(A) = A^n \quad \text{for all } n \in \mathbb{N}_0, \qquad \text{where } p_n(\lambda) := \lambda^n,$$

i.e. Ψ is a functional calculus for the operator A for functions that are holomorphic in a neighborhood of $\sigma(A)$.

Proof. (1) is clear. (2): Linearity is clear. The proof of multiplicativity is similar to what we have done in the proof of 9.1(a). This implies

$$p_n(A) = \frac{1}{2\pi i}\int_\Gamma \lambda^n R(\lambda,A), d\lambda = p_1(A)^n, \quad n \in \mathbb{N}_0.$$

Moreover, $Ap_0(A) = p_1(A)p_0(A)$. Thus it rests to show $p_0(A) = I_X$. Apply 9.1 with $\sigma_0 = \sigma(A)$ and $\sigma_1 = \emptyset$. Then $A_1 \in \mathcal{L}(X_1)$ (since $A \in \mathcal{L}(X)$), and $\sigma(A_1) = \sigma_1 = \emptyset$. Hence $X_1 = \{0\}$, $X_0 = X$, and $p_0(A) = P = I_X$. $\square$

9.4. Remark: Let $A \in \mathcal{L}(X)$ and U be an open neighborhood of $\overline{B}(0, r(A)) \supseteq \sigma(A)$. In this case we can choose $\Gamma = \partial B(0, r)$ for a suitable $r > r(A)$. By 8.9 and its proof we know

$$R(\lambda, A) = \sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}}, \quad |\lambda| > r(A).$$

If $f : U \to \mathbb{C}$ is holomorphic then we obtain

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda) R(\lambda, A)\, d\lambda = \frac{1}{2\pi i} \int_\Gamma \sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}}\, d\lambda = \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} A^k,$$

where we interchanged summation and integration by uniform convergence and used Cauchy's integral formula. This means that we get $f(A)$ for such f by inserting A for z in the power series expansion of f around 0. Of course, then $p_n(A) = A^n$ for $n \in \mathbb{N}_0$ is clear.

10 Fourier transform on $\mathbb{R}^d$

10.1. Definition: For $f \in L^1(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$ we define

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x\xi} f(x)\, dx,$$

where $x\xi = x_1\xi_1 + x_2\xi_2 + \ldots + x_d\xi_d$ is the scalar product in $\mathbb{R}^d$. The function $\hat{f} : \mathbb{R}^d \to \mathbb{C}$ is called the *Fourier transform of f*.

10.2. Lemma: (a) For $f \in L^1(\mathbb{R}^d)$ the function $\hat{f}$ is bounded and uniformly continuous and

$$\|\hat{f}\|_\infty \leq \|f\|_1.$$

The map $f \to \hat{f}$ is linear and continuous $L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$.

(b) Let $f \in L^1(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$. For $\tau_y f := f(\cdot - y)$ we have $\widehat{\tau_y f}(\xi) = e^{-2\pi i y\xi}\hat{f}(\xi)$, $\xi \in \mathbb{R}^d$.

(c) For $f, g \in L^1(\mathbb{R}^d)$ and $\lambda > 0$ we have $\int \hat{f}(\xi)g(\lambda\xi)\, d\xi = \int f(\lambda x)\hat{g}(x)\, dx$.

Proof. (a) The estimate is clear. For $\xi, \eta \in \mathbb{R}^d$ we have

$$\begin{aligned} |\hat{f}(\xi) - \hat{f}(\xi + \eta)| &\leq \int |e^{-2\pi i x\xi} - e^{-2\pi i x(\xi+\eta)}|\, |f(x)|\, dx \\ &= \int |1 - e^{-2\pi i x\eta}|\, |f(x)|\, dx \text{ indep. of } \xi. \end{aligned}$$

By $1 - e^{-2\pi i x\eta} \to 0$ $(\eta \to 0)$ for fixed $x \in \mathbb{R}^d$ and $|1 - e^{-2\pi i x\eta}| \leq 2$ the integral converges to 0 as $\eta \to 0$ by dominated convergence. Hence $\hat{f}$ is uniformly continuous.

(b) We substitute $x = u + y$ and obtain

$$\int e^{-2\pi i x\xi} f(x - y)\, dx = e^{-2\pi i y\xi} \int e^{-2\pi i u\xi} f(u)\, du = e^{-2\pi i y\xi}\hat{f}(\xi).$$

(c) We substitute $x = \lambda y$ and $\lambda\xi = \eta$ and obtain by Fubini

$$\int\int e^{-2\pi i x\xi} f(x)g(\lambda\xi)\, dx\, d\xi = \int f(\lambda y) \underbrace{\int e^{-2\pi i y\eta} g(\eta)\, d\eta}_{=\hat{g}(y)}\, dy.$$

□

10.3. Example: Let $d = 1$ and $h = 1_{[a,b]}$. Then $\hat{h}(0) = b - a$ and for $\xi \neq 0$ we have

$$\hat{h}(\xi) = \int_a^b e^{-2\pi i x\xi}\, dx = \frac{e^{-2\pi i b\xi} - e^{-2\pi i a\xi}}{-2\pi i\xi}.$$

Hence $\hat{h}(\xi) \to 0$ ($|\xi| \to \infty$), but $\hat{h} \notin L^1(\mathbb{R})$!

For $d > 1$ and $h(x) = 1_{\prod_{j=1}^n [a_j,b_j]}(x) = \prod_{j=1}^n 1_{[a_j,b_j]}(x_j)$ we obtain

$$\hat{h}(\xi) = \prod_{j=1}^n \widehat{1_{[a_j,b_j]}}(\xi_j), \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^d.$$

Hence also here we have $\hat{h}(\xi) \to 0$ für $|\xi| \to \infty$.

10.4. Riemann-Lebesgue-Lemma: For each $f \in L^1(\mathbb{R}^d)$ we have $\hat{f}(\xi) \to 0$ für $|\xi| \to \infty$.

Proof. The space

$$C_0(\mathbb{R}^d) := \{ f : \mathbb{R}^d \to \mathbb{C} \text{ stetig} : f(x) \to 0 \ (|x| \to \infty) \}$$

is a closed linear subspace w.r.t. the norm $\|\cdot\|_\infty$ of the Banach space

$$BUC(\mathbb{R}^d) := \{ g : \mathbb{R}^d \to \mathbb{C} : \ g \text{ is bounded and uniformly continuous} \}.$$

By Example 10.3 we have $\hat{h} \in C_0(\mathbb{R}^d)$ for functions h of the form $h = \sum_k c_k 1_{Q_k}$, where the Q_k are cartesian products of compact intervals. The set of these simple functions is dense in $L^1(\mathbb{R}^d)$, and we employ 10.2 (a). □

10.5. Rules: Let $f \in L^1(\mathbb{R}^d)$. Then the following hold:

(a) If $a \in \mathbb{R} \setminus \{0\}$ then $\mathscr{F}(F(a\cdot))(\xi) = \frac{1}{|a|^n} \hat{f}(\frac{\xi}{a})$, $\xi \in \mathbb{R}^d$.

(b) If $b \in \mathbb{R}^d$ then $\mathscr{F}(e^{2\pi i b(\cdot)} f)(\xi) = \hat{f}(\xi - b)$, $\xi \in \mathbb{R}^d$.

(c) If in addition $x \mapsto x_j f(x) \in L^1(\mathbb{R}^d)$ then $\hat{f}$ has a continuous partial derivative w.r.t ξ_j and

$$\frac{\partial}{\partial \xi_j} \hat{f}(\xi) = \mathscr{F}(x \mapsto (-2\pi i x_j) f(x))(\xi), \quad \xi \in \mathbb{R}^d.$$

Proof. The proof of (a) is similar to the one for 10.2(c). For the proof of (b) we write

$$\int \underbrace{e^{-2\pi i \xi x} e^{2\pi i b x}}_{= e^{-2\pi i (\xi - b) x}} f(x)\, dx = \hat{f}(\xi - b).$$

ad (c): For $\xi \in \mathbb{R}^d$, $j \in \{1, \ldots, d\}$ and $h \neq 0$ we have

$$\begin{aligned} &\frac{\hat{f}(\xi + h e_j) - \hat{f}(\xi)}{h} - \mathscr{F}(x \mapsto (-2\pi i x_j) f(x))(\xi) \\ = &\int \Big(\frac{e^{-2\pi i (\xi + h e_j) x} - e^{-2\pi i \xi x}}{h} + 2\pi i x_j e^{-2\pi i \xi x} \Big) f(x)\, dx. \end{aligned}$$

For fixed $x \in \mathbb{R}^d$ the term in $(\ldots)$ tends to 0 as $h \to 0$. Moreover we have

$$\begin{aligned} e^{-2\pi i \xi x}\Big(\frac{e^{-2\pi i h x_j} - 1}{h} + 2\pi i x_j\Big) &= e^{-2\pi i \xi x}\Big(\frac{1}{h}\int_0^h (-2\pi i x_j) e^{-2\pi i t x_j}\, dt + 2\pi i x_j\Big) \\ &= e^{-2\pi i \xi x} 2\pi i x_j \Big(1 - \frac{1}{h}\int_0^h e^{-2\pi i t x_j}\, dt\Big). \end{aligned}$$

Here the last term in $(\ldots)$ is ≤ 2 in modulus. The assertion follows by dominated convergence. □

end Thu 09.05.19

10.6. Proposition (weak derivatives and Fourier transform): Let $f \in L^1(\mathbb{R}^d)$ and let $g \in L^1(\mathbb{R}^d)$ such that

$$\int f(x)\partial_j \varphi(x)\, dx = -\int g(x)\varphi(x)\, dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, i.e. f has weak partial derivative $g = \partial_j f$. Then

$$\hat{g}(\xi) = 2\pi i \xi_j \hat{f}(\xi), \quad \xi \in \mathbb{R}^d.$$

Remark: The assumption is in particular satisfied if f is continuously differentiable with $g := \partial_j f \in L^1(\mathbb{R}^d)$.

Proof. Let $\xi \in \mathbb{R}^d$. The idea is to take $\varphi(x) = e^{-2\pi i x \xi}$, but $\varphi \notin C_c^\infty$. Thus we approximate this function. we choose $\psi \in C_c^\infty$ with $0 \leq \psi \leq 1$ and $\psi(x) = 1$ for $|x| \leq 1$ and set $\psi_k(x) := \psi(x/k)$ and $\varphi_k(x) := e^{-2\pi i x \xi}\psi_k(x)$. Dominated convergence yields

$$\int g\varphi_k\, dx = \int g(x) e^{-2\pi i x \xi}\psi_k(x)\, dx \to \hat{g}(\xi)$$

since $\psi_k(x) \to 1$ for each $x \in \mathbb{R}^d$ and $|\psi_k| \leq 1$. Moreover we have, by $\varphi_k \in C_c^\infty$,

$$\int g\varphi_k\, dx = -\int f\partial_j\varphi_k\, dx = \underbrace{-\int f(-2\pi i \xi_j)\varphi_k\, dx}_{\to 2\pi i \xi_j \hat{f}(\xi)} - \underbrace{\int f(x) e^{-2\pi i x \xi}\partial_j\psi_k(x)\, dx}_{=:A(k)}.$$

Here $\partial_j \psi_k(x) = \frac{1}{k}(\partial_j\psi)(x/k)$ and thus

$$|A(k)| \leq \int_{|x| \geq k} |f(x)|\, dx \cdot \frac{1}{k} \cdot \|\partial_j \psi\|_\infty \to 0 \ (k \to \infty),$$

since $\partial_j \psi$ is bounded. □

10.7. Example: For $\phi(x) = e^{-\pi|x|^2}$, $x \in \mathbb{R}^d$, we have

$$\hat{\phi}(\xi) = e^{-\pi|\xi|^2}, \quad \xi \in \mathbb{R}^d.$$

By $\phi(x) = \prod_{j=1}^n e^{-\pi|x_j|^2}$ it suffices to study the case $d = 1$. Then we have for each $x \in \mathbb{R}$:

$$\phi'(x) = -2\pi x \phi(x).$$

By 10.5(c) and 10.6 we obtain for each $\xi \in \mathbb{R}$:

$$2\pi i \xi \hat{\phi}(\xi) = \widehat{\phi'}(\xi) = -\widehat{2\pi x \phi(x)}(\xi) = -i\frac{d}{d\xi}\hat{\phi}(\xi),$$

hence

$$(\hat{\phi})'(\xi) = -2\pi\xi\hat{\phi}(\xi).$$

This means that ϕ and $\hat{\phi}$ solve the same homogeneous linear differential equation. By uniqueness of solutions of the initial value problem we thus have

$$\hat{\phi}(\xi) = \hat{\phi}(0)\phi(\xi) = e^{-\pi|\xi|^2}, \quad \xi \in \mathbb{R},$$

since $\hat{\phi}(0) = \int_{\mathbb{R}} e^{-\pi|x|^2}\, dx = 1$.

10.8. Fourier inversion formula: Let $f \in L^1(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$ such that $\hat{f} \in L^1(\mathbb{R}^d)$. Then we have for any $x \in \mathbb{R}^d$:

$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\xi} \hat{f}(\xi)\, d\xi.$$

Remark: Actually, the formula holds in an a.e.-sense for all $f \in L^1(\mathbb{R}^d)$ with $\hat{f} \in L^1(\mathbb{R}^d)$. ($\rightarrow$ below).

Proof. By 10.2(b) it suffices to study $x = 0$. Let $h(x) := e^{-\pi|x|^2}$, $x \in \mathbb{R}^d$. By 10.2(c) we have for any $a > 0$:

$$\int_{\mathbb{R}^d} \hat{f}(\xi)h(a\xi)\, d\xi = \int_{\mathbb{R}^d} f(ax)\hat{h}(x)\, dx.$$

As $a \to 0+$, the left hand side tends to $\int \hat{f}(\xi)\, d\xi$ (use dominated convergence and the assumption $\hat{f} \in L^1$). Since f is continuous at 0, the integrand in the right hand side converges pointwise to $f(0)\hat{h}(x)$. Since f is bounded, dominated convergence yields that the right hand side tends to $f(0) \int \hat{h}(x)\, dx = f(0)$. □

10.9. Example: $f(x) = e^{-a|x|}$ for $x \in \mathbb{R}$ where $a > 0$. We have

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi x} e^{-a|x|}\, dx$$

and

$$\int_0^\infty e^{-2\pi i x\xi} e^{-ax}\, dx = \frac{1}{a + 2\pi i \xi}, \quad \int_{-\infty}^0 e^{-2\pi i x\xi} e^{ax}\, dx = \int_0^\infty e^{2\pi i y\xi} e^{-ay}\, dy = \frac{1}{a - 2\pi i \xi}.$$

Hence

$$\hat{f}(\xi) = \frac{1}{a + 2\pi i \xi} + \frac{1}{a - 2\pi i \xi} = \frac{2a}{a^2 + 4\pi^2 \xi^2}, \quad \xi \in \mathbb{R}.$$

The functions f is bounded and continuous and $\hat{f} \in L^1(\mathbb{R})$. Hence we have by 10.8:

$$\mathscr{F}\Big(x \mapsto \frac{2a}{a^2 + 4\pi^2 x^2}\Big)(\xi) = e^{-a|\xi|}, \quad \xi \in \mathbb{R}.$$

10.10. Definition and Proposition (Convolution): Let $f, g \in L^1(\mathbb{R}^d)$. For a.e. $x \in \mathbb{R}^d$ we have $y \mapsto f(y)g(x-y) \in L^1(\mathbb{R}^d)$, and for the function h, given by

$$h(x) := \int_{\mathbb{R}^d} f(y)g(x-y)\, dy \text{ for a.e. } x \in \mathbb{R}^d,$$

we have $h \in L^1(\mathbb{R}^d)$ and

$$\|h\|_1 \leq \|f\|_1 \|g\|_1.$$

The function h is called *convolution of f and g* and is denoted by $f * g$.

Remark: In FA 5.2 we defined the convolution of L^1_{loc}-functions with C_c^∞-functions. It is clear that this coincides with the convolution defined here if the L^1_{loc}-function is actually in L^1.

Proof. The function $F : (x, y) \mapsto f(y)g(x-y)$ is measurable and

$$\int\int |f(y)g(x-y)|\, dx\, dy = \int |f(y)| \underbrace{\int |g(x-y)|\, dx}_{=\|g\|_1}\, dy = \|f\|_1 \|g\|_1.$$

By Fubini-Tonelli we thus have $F \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$, $F(x, \cdot) \in L^1(\mathbb{R}^d)$ for a.e. $x \in \mathbb{R}^d$, and $x \mapsto \int F(x, y)\, dy \in L^1(\mathbb{R}^d)$. □

10.11. Lemma: The convolution $* : L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is bilinear and continuous, commutative, and associative. For $f, g \in L^1(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$ we have

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

Proof. Bilinearity and continuity are clear by 10.10. Commutativity and associativeness follow by Fubini. We show the formula. By Fubini we have

$$\begin{aligned}\int e^{-2\pi i x\xi} f * g(x)\,dx &= \int\int e^{-2\pi i x\xi} f(y)g(x-y)\,dx\,dy \\ &= \int e^{-2\pi i y\xi} f(y) \underbrace{\int e^{-2\pi i(x-y)\xi} g(x-y)\,dx}_{=\hat{g}(\xi)}\,dy = \hat{f}(\xi)\hat{g}(\xi).\end{aligned}$$

□

10.12. Lemma: For $f, \hat{f} \in L^1(\mathbb{R}^d)$ we have the Fourier inversion formula 10.8. In particular, we then have $f, \hat{f} \in C_0(\mathbb{R}^d)$.

Proof. is an exercise. □

10.13. Theorem (Plancherel): For $f \in L^1(\mathbb{R}^d)$ with $\hat{f} \in L^1(\mathbb{R}^d)$ we have $f, \hat{f} \in L^2(\mathbb{R}^d)$ and

$$\|f\|_2^2 = \|\hat{f}\|_2^2.$$

Proof. By $\|g\|_2^2 \leq \|g\|_\infty \|g\|_1$ we get from 10.12: $f, \hat{f} \in L^2(\mathbb{R}^d)$. By 10.2 (dancing hat) we have

$$\int_{\mathbb{R}^n} |\hat{\psi}(\xi)|^2\,d\xi = \int_{\mathbb{R}^n} \hat{\psi}(\xi)\overline{\hat{\psi}(\xi)}\,d\xi = \int_{\mathbb{R}^n} \psi(x)(\overline{\hat{\psi}})^\wedge(x)\,dx,$$

and by 10.8 we get

$$(\overline{\hat{\psi}})^\wedge(x) = \int_{\mathbb{R}^n} e^{-2\pi i\xi x}\overline{\hat{\psi}(\xi)}\,d\xi = \overline{\int_{\mathbb{R}^n} e^{2\pi i\xi x}\hat{\psi}(\xi)\,d\xi} = \overline{\psi(x)}.$$

□

end Tue 14.05.19

10.14. Theorem: Let $S := \{f \in L^1(\mathbb{R}^d) : \hat{f} \in L^1(\mathbb{R}^d)\}$. The Fourier transform $S \to S$, $f \mapsto \hat{f}$, has a unique continuous extension $\mathscr{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. This extension $\mathscr{F}$ is linear and bijective, and we have

$$\|\mathscr{F}f\|_2^2 = \|f\|_2^2, \quad f \in L^2(\mathbb{R}^d),$$

and

$$\mathscr{F}^{-1}f = \sigma\mathscr{F}f, \quad f \in L^2(\mathbb{R}^d), \qquad \mathscr{F}^4 = \mathrm{Id}_{L^2}.$$

Moreover, we have for all $f, g \in L^2(\mathbb{R}^d)$:

$$(f|g)_{L^2} = (\hat{f}|\hat{g})_{L^2}, \quad \text{i.e. } \int_{\mathbb{R}^d} f(x)\overline{g(x)}\,dx = \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi.$$

Proof. We only have to show that S is dense in $L^2(\mathbb{R}^d)$. To see this, it suffices to approximate $f \in L^2(\mathbb{R}^n)$ with compact support by functions in S, which can be done with mollifiers: Let $\phi(x) = e^{-\pi|x|^2}$, $\phi_k(x) = k^d\phi(k\cdot)$, and $f_k := f * \phi_k$. Then $f_k \in L^1$, $\hat{f}_k = \hat{f}\hat{\phi}_k \in L^1$ (since $\hat{f} \in BUC$) and $f_k \to f$ in $\|\cdot\|_{L^2}$ (this is shown as in FA). □

10.15. Definition: The *Schwartz space* $\mathscr{S}(\mathbb{R}^d)$ is given by

$$\mathscr{S}(\mathbb{R}^d) := \{\psi : \mathbb{R}^d \to \mathbb{C} : \ \psi \text{ is } C^\infty \text{ and } \forall \alpha, \beta \in \mathbb{N}_0^d : x \mapsto x^\beta \partial^\alpha \psi(x) \text{ bounded}\},$$

where we use multi-index notation: For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d), \beta = (\beta_1, \beta_2, \ldots, \beta_d) \in \mathbb{N}_0^d$ we have

$$\begin{aligned}
\partial^\alpha &:= \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}, \\
|\alpha| &:= \alpha_1 + \alpha_2 + \ldots + \alpha_d, \\
\alpha! &:= \alpha_1!\alpha_2! \cdot \ldots \cdot \alpha_d!, \\
\beta \le \alpha &:\Leftrightarrow \beta_1 \le \alpha_1, \beta_2 \le \alpha_2, \ldots, \beta_d \le \alpha_d, \\
\binom{\alpha}{\beta} &:= \frac{\alpha!}{\beta!(\alpha-\beta)!} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \cdot \ldots \cdot \binom{\alpha_d}{\beta_d}, \beta \le \alpha, \\
x^\alpha &:= x_1^{\alpha_1} x_2^{\alpha_2} \cdot \ldots \cdot x_d^{\alpha_d}, x = (x_1, x_2, \ldots, x_d) \in \mathbb{C}^d.
\end{aligned}$$

Remark: We recall the *Leibniz rule* for products

$$\partial^\alpha(\varphi \cdot \psi) = \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} \partial^\gamma \varphi \partial^{\alpha-\gamma} \psi,$$

which can be hown by induction on $|\alpha|$.

Remark: For $\psi \in \mathscr{S}(\mathbb{R}^d)$ and any $\alpha \in \mathbb{N}_0^d$ we have that $\partial^\alpha \psi$ and $x \mapsto x^\alpha \psi(x)$ belong to $\mathscr{S}(\mathbb{R}^d)$.

10.16. Proposition: For $\varphi, \psi \in \mathscr{S}(\mathbb{R}^d)$ we have $\varphi \cdot \psi \in \mathscr{S}(\mathbb{R}^d)$.

Proof. Use the Leibniz rule

$$x^\beta \partial^\alpha(\varphi \cdot \psi) = \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} \underbrace{x^\beta \partial^\gamma \varphi}_{\text{bounded}} \underbrace{\partial^{\alpha-\gamma} \psi}_{\text{bounded}}.$$

□

10.17. Theorem: The Fourier transform $\psi \mapsto \hat{\psi}$ is linear and bijective $\mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$.

Proof. is an exercise. □

10.18. Proposition: For $\varphi, \psi \in \mathscr{S}(\mathbb{R}^d)$ we have $\varphi * \psi \in \mathscr{S}(\mathbb{R}^d)$.

Proof. By $\widehat{\varphi * \psi} = \hat{\varphi} \cdot \hat{\psi}$ this follows from 10.17 and 10.16. $\square$

10.19. Sobolev spaces via Fourier transform: First we show that, for any $f \in L^2(\mathbb{R}^d)$ and $j \in \{1, \ldots, d\}$, we have

$$f \text{ has a partial weak derivative } \partial_j f \in L^2(\mathbb{R}^d) \iff \xi \mapsto \xi_j \hat{f}(\xi) \in L^2(\mathbb{R}^d),$$

and in this case $\widehat{\partial_j f}(\xi) = 2\pi i \xi_j \hat{f}(\xi)$ for a.e. $\xi \in \mathbb{R}^d$. This is an L^2-version of 10.5(c) and 10.6.

Proof. Let f have partial weak derivative $\partial_j f \in L^2(\mathbb{R}^d)$. For $\varphi \in C_c^\infty(\mathbb{R}^d)$ we then have by 10.14 and 10.5(c)

$$\int (\partial_j f)\, \overline{\varphi}\, dx = -\int f\, \overline{\partial_j \varphi}\, dx = 2\pi i \int \hat{f}\, \xi_j \overline{\hat{\varphi}}\, d\xi.$$

This implies

$$\Big| \int \hat{f} \xi_j \overline{\hat{\varphi}}\, d\xi \Big| \leq \frac{1}{2\pi} \|\partial_j f\|_{L^2} \|\varphi\|_{L^2},$$

which in turn yields $\xi_j \hat{f} \in L^2(\mathbb{R}^d)$ (we again use Plancherel here). If, on the other hand, $\xi_j \hat{f} \in L^2(\mathbb{R}^d)$ then we have $g := \mathscr{F}^{-1}(2\pi i \xi_j \hat{f}(\xi)) \in L^2(\mathbb{R}^d)$ and obtain, for any $\varphi \in C_c^\infty$,

$$\int g \overline{\varphi}\, dx = \int \hat{g} \overline{\hat{\varphi}}\, d\xi = 2\pi i \int \hat{f} \xi_j \overline{\hat{\varphi}}\, d\xi = -\int \hat{f}\, \overline{\widehat{\partial_j \varphi}}\, d\xi = -\int f (\partial_j \varphi)\, dx.$$

But this means $g = \partial_j f$ as a weak derivative. Observe $\hat{g}(\xi) = 2\pi i \xi_j \hat{f}(\xi)$. $\square$

For $s \geq 0$ we define

$$H^s(\mathbb{R}^d) := \{ f \in L^2(\mathbb{R}^d) : \xi \mapsto (1 + 4\pi^2 |\xi|^2)^{s/2} \hat{f}(\xi) \in L^2(\mathbb{R}^d) \}$$

and

$$\|f\|_{H^s} := \|\xi \mapsto (1 + 4\pi^2|\xi|^2)^{s/2} \hat{f}(\xi)\|_{L^2} = \Big(\int_{\mathbb{R}^d} (1 + 4\pi^2 |\xi|^2)^s |\hat{f}(\xi)|^2\, d\xi \Big)^{1/2}.$$

Then $H^s(\mathbb{R}^d)$ is a Hilbert space, where the scalar product is given by

$$(f|g)_{H^s} = \int_{\mathbb{R}^d} (1 + 4\pi^2 |\xi|^2)^s \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, d\xi.$$

We claim that $H^m(\mathbb{R}^d) = W^{m,2}(\mathbb{R}^d)$ for any $m \in \mathbb{N}$ with equivalent norms. This is proved via the following observation which follows from Plancherel:

Let $m : \mathbb{R}^d \to \mathbb{C}$ be measurable. Then $T_m f := \mathscr{F}^{-1}(m\hat{f})$ defines a bounded linear operator on $L^2(\mathbb{R}^d)$ if and only if $m \in L^\infty(\Omega)$.

Proof. Let $m \in \mathbb{N}$. Recall

$$\begin{aligned} W^{m,2}(\mathbb{R}^d) &:= \{f \in L^2(\mathbb{R}^d) : \forall \alpha \in \mathbb{N}_0^d : \partial^\alpha f \in L^2(\mathbb{R}^d)\} \\ &= \{f \in W^{m-1,2}(\mathbb{R}^d) : \forall j \in \{1,\dots,d\} : \partial_j f \in W^{m-1,2}(\mathbb{R}^d)\} \end{aligned}$$

It thus suffices to show $H^1 = W^{1,2}$ (then one can use a simple induction). If $f \in H^1$ then also

$$\xi \mapsto \xi_j \hat{f}(\xi) = \underbrace{\frac{\xi_j}{(1+4\pi^2|\xi|^2)^{1/2}}}_{\text{bounded}} (1+4\pi^2|\xi|^2)^{1/2}\hat{f}(\xi) \in L^2(\mathbb{R}^d).$$

If $\partial_j f \in L^2$ for all j then $\xi_j \hat{f} \in L^2$ for all j (as we have seen above). By

$$(1+|\xi|^2)^{1/2} \le 1 + |\xi| \le 1 + \sum_j |\xi_j|$$

and the observation we conclude that $f \in H^1$. The estimates also show equivalence of norms. □

10.20. The Laplace operator in $L^2(\mathbb{R}^d)$: We define the Laplace operator Δ on $L^2(\mathbb{R}^d)$ by $\Delta f := \sum_{j=1}^d \partial_j^2 f$ for $f \in D(\Delta) := W^{2,2}(\mathbb{R}^d) = H^2(\mathbb{R}^d)$.

Then $-\Delta = \mathscr{F}^{-1} M_m \mathscr{F}$ where M_m is the *multiplication operator* given by $M_m f := mf$ with $m(\xi) = 4\pi^2|\xi|^2$, which has domain $D(M_m) = \{f \in L^2(\mathbb{R}^d) : mf \in L^2(\mathbb{R}^d)\}$ (here we use 10.19). Since the Fourier transform is a unitary operator on $L^2(\mathbb{R}^d)$ this means that Δ is *unitarily equivalent to a multiplication operator*.

Since M_m is closed, we conclude by 8.4 that $-\Delta$ is closed. But we also obtain that $\sigma(-\Delta) = [0,\infty)$ and that

$$R(\lambda,-\Delta)f = (\lambda+\Delta)^{-1}f = \mathscr{F}^{-1}\big(\xi \mapsto \underbrace{(\lambda - 4\pi^2|\xi|^2)^{-1}}_{\in L^\infty}\hat{f}(\xi)\big), \quad f \in L^2(\mathbb{R}^d),$$

for any $\lambda \in \mathbb{C} \setminus [0,\infty)$.

Moreover, we can easily define a *functional calculus* for $-\Delta$ by

$$L^\infty(0,\infty) \to \mathcal{L}(L^2(\mathbb{R}^d)), \quad F \mapsto F(-\Delta) := T_{F\circ m} = \mathscr{F}^{-1} M_{F\circ m}\mathscr{F}.$$

i.e. by letting

$$F(-\Delta)f = \mathscr{F}^{-1}\big(\xi \mapsto F(4\pi^2|\xi|^2)\hat{f}(\xi)\big).$$

This is an algebra homomorphism, i.e. $F \mapsto F(-\Delta)$ is linear and $(FG(-\Delta) = F(-\Delta)G(-\Delta)$, and we have

$$\|F(-\Delta)\|\mathcal{L}(L^2(\mathbb{R}^d)) = \|\xi \mapsto F(4\pi^2|\xi|^2)\|_{L^\infty(\mathbb{R}^d)} = \|F\|_{L^\infty(0,\infty)}.$$

We observe that $F(-\Delta) = I$ for $F(z) = 1$ and $R(\lambda, -\Delta) = F_\lambda(-\Delta)$ where $F_\lambda(z) = (\lambda - z)^{-1}$. We also observe that $F(-\Delta)$ is self-adjoint if F is real-valued:

$$(F(-\Delta)f|g) = (F \circ m \cdot \hat{f}|\hat{g}) = (\hat{f}|F \circ m \cdot \hat{g}) = (f|F(-\Delta)g).$$

The functional calculus for $-\Delta$ is but a special case of the general spectral theorem for unbounded self-adjoint operators that we shall study lateron.

end Thu 16.05.19

10.21. Theorem (Sobolev embedding): For $s > d/2$ and $\gamma \in (0, s - d/2) \cap (0, 1)$ the space $H^s(\mathbb{R}^d)$ is continuously embedded into the space of Hölder-continuous functions

$$C^\gamma(\mathbb{R}^d) = \{f \in C(\mathbb{R}^d) : \|f\|_\infty < \infty, \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma} < \infty\}.$$

Remark: The space $C^\delta(\mathbb{R}^d)$ is a Banach space w.r.t. the norm

$$\|f\|_{C^\delta} := \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\delta},$$

where $\delta \in (0, 1]$.

Proof. For $f \in H^s(\mathbb{R}^d)$ we write

$$\mathscr{F} f(\xi) = \underbrace{(1 + 4\pi^2|\xi|^2)^{-s/2}}_{\in L^2} \underbrace{(1 + 4\pi^2|\xi|^2)^{s/2} \mathscr{F} f(\xi)}_{\in L^2} \in L^1,$$

from which we get by Fourier inversion that $f \in C_0(\mathbb{R}^d)$ and that

$$\|f\|_\infty \leq C_1 \|f\|_{H^s(\mathbb{R}^d)}.$$

Moreover, we have $|\xi|^\gamma \mathscr{F} f(\xi) \in L^1$ and

$$\|\xi \mapsto |\xi|^\gamma \mathscr{F} f(\xi)\|_{L^1(\mathbb{R}^d)} \leq C_2 \|f\|_{H^s(\mathbb{R}^d)}.$$

Now we write

$$f(x) - f(y) = \int_{\mathbb{R}^d} (e^{2\pi i x \cdot \xi} - e^{2\pi i y \cdot \xi}) \mathscr{F}(\xi) \, d\xi$$

and obtain by

$$|e^{2\pi i x \cdot \xi} - e^{2\pi i y \cdot \xi}| \leq \min\{2, c|x - y||\xi|\} \leq C|x - y|^\gamma |\xi|^\gamma,$$

that

$$|f(x) - f(y)| \leq C'|x - y|^\gamma \underbrace{\||\cdot|^\gamma \mathscr{F} f\|_{L^1}}_{\leq C_2 \|f\|_{H^s}}.$$

We thus have proved the continuous embedding $H^s(\mathbb{R}^d) \subseteq C^\gamma(\mathbb{R}^d)$ for $\gamma \in (0, \min\{1, s - d/2\})$. □

10.22. Theorem (Hausdorff-Young): Let $p \in [1, 2]$ and let p' be the dual exponent, given by $\frac{1}{p} + \frac{1}{p'} = 1$. Then the Fourier transform $\mathscr{F}$ is a bounded operator $L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$.

Proof. By 10.2(a) the assertion holds for $p = 1$. By 10.14 it holds for $p = 2$. We obtain it for $p \in (1, 2)$ by Riesz-Thorin interpolation ($\to$ FA). □

11 Fredholm operators and the spectral theory of compact operators

Again, X and Y are Banach spaces if nothing else is said.

Recall from Linear Algebra: If $\dim X = \dim \mathrm{Y} < \infty$ and $T \in \mathcal{L}(X,Y)$ then

$$\dim N(T) = \underbrace{\operatorname{codim} R(T)}_{=:\dim Y/R(T)} < \infty.$$

11.1. Definition: An operator $T \in \mathcal{L}(X,Y)$ is called a *Fredholm operator* if

$$\alpha(T) := \dim N(T) < \infty \text{ and } R(T) \text{ is closed with } \beta(T) := \operatorname{codim} R(T) < \infty.$$

In this case, the *index of* T is defined by

$$\operatorname{ind} T := \alpha(T) - \beta(T).$$

The set of all Fredholm operators from X to Y is denoted by $\Phi(X,Y)$.

Remark: If $\dim X = \dim Y < \infty$, any $T \in \mathcal{L}(X,Y)$ is a Fredholm operator of index 0. If X and Y are arbitrary and $T \in \mathcal{L}(X,Y)$ is an isomorphism from X to Y then $\alpha(T) = \beta(T) = 0$ and T is a Fredholm operator of index 0.

The following proposition shows that closedness of $R(T)$ can be omitted from the definition.

11.2. Proposition: Let $T \in \mathcal{L}(X,Y)$. If $\operatorname{codim} R(T) < \infty$ then $R(T)$ is closed.

Proof. We find a basis $[y_1], \dots, [y_n]$ of $Y/R(T)$ and let $H := \operatorname{span}\{y_1, \dots, y_n\}$. Then $Y = R(T) + H$ and $R(T) \cap H = \{0\}$. Since H is complete, $X \times H$ is a Banach space (e.g. for the sum-norm) and the map

$$\widetilde{T} : X \times H \to Y,\ (x,y) \mapsto Tx + y$$

is continuous and surjective. By the Open Mapping Theorem we have

$$\gamma := \inf \left\{ \frac{\|\widetilde{T}(x,y)\|_Y}{d((x,y), N(\widetilde{T}))} : (x,y) \notin N(\widetilde{T}) \right\} > 0.$$

Since

$$N(\widetilde{T}) = \{(x,y) \in X \times H : Tx = -y\} = \{(x,0) : Tx = 0\} = N(T) \times \{0\}$$

we have for $x \notin N(T)$:

$$\|Tx\|_Y = \|\widetilde{T}(x,0)\|_Y \geq \gamma d((x,0), N(\widetilde{T})) = \gamma d(x, N(T)).$$

From this we obtain closedness of $R(T)$. □

Remark: We recall that, if Y is a Banach space and Z is a closed linear subspace, then the *quotient space* Y/Z is a Banach space defined by

$$Y/Z := \{y + Z : y \in Y\}, \|y + Z\|_{Y/Z} := \inf\{\|y - z\|_Y : z \in Z\} = d(y, Z).$$

If Z is clear from the context, it is common to write $[y]$ instead of $y + Z$ and q for the *quotient map* $Y \to Y/Z$, $y \mapsto [y]$. We also recall that an operator $T \in \mathcal{L}(X, Y)$ has a unique factorization $T = S \circ q$ where $S \in \mathcal{L}(X/N(T), Y)$ is injective and $q : X \to X/N(T)$ is the quotient map. S is given by $S[x] = Tx$.

Remark: (a) Suppose that W and Z are subspaces of a Banach space X such that $W \cap Z = \{0\}$ and $X = W + Z$. We write $X = W \oplus Z$ if both W and Z are closed subspaces of X. Notice that $X = W \oplus Z$ if and only if $W \times Z \to X$, $(w, z) \mapsto w + z$, is an isomorphism if and only if there exists a projection $P \in \mathcal{L}(X)$ such that $P(X) = Z$ and $N(P) = W$.

A closed subspace Z is called *complemented* in X if there is a (closed) subspace W of X such that $X = W \oplus Z$.

(b) Any finite-dimensional subspace Z of X is complemented in X: Choose a basis $z_1, \dots, z_n$ of Z and linear functionals $\psi_1, \dots, \psi_n \in Z'$ such that $\psi_j(x_k) = \delta_{jk}$. Extend the ψ_k by Hahn-Banach and obtain $\phi_1, \dots, \phi_n \in X'$. Then $W := \bigcap_{k=1}^n N(\phi_k)$ is a complement of Z in X.

11.3. Theorem: $\Phi(X, Y)$ is open in $\mathcal{L}(X, Y)$ and $\operatorname{ind} : \Phi(X, Y) \to \mathbb{Z}$ is continuous.

Proof. Let $T \in \Phi(X, Y)$. We shall show the existence of $\varepsilon > 0$ such that, for $S \in \mathcal{L}(X, Y)$, we have

$$\|S - T\| < \varepsilon \implies S \in \Phi(X, Y) \text{ and } \operatorname{ind} S = \operatorname{ind} T.$$

First we find closed subspaces G of X and H of Y such that

$$X = N(T) \oplus G, \; Y = R(T) \oplus H, \; \dim H < \infty.$$

Notice that $R(T) = T(X) = T(G)$. For any $S \in \mathcal{L}(X, Y)$ we define

$$\widehat{S} : G \times H \to Y, \quad (g, h) \mapsto Sg + h.$$

Then we have, for $S_1, S_2 \in \mathcal{L}(X, Y)$:

$$\|\widehat{S_1} - \widehat{S_2}\| = \sup_{\|(g,h)\|=1} \|S_1 g + h - (S_2 g + h)\| \le \|S_1 - S_2\|.$$

We claim that $\widehat{T} : G \times H \to Y$ is an isomorphism. Indeed, $\widehat{T}$ is surjective by $R(T) = T(G)$ and the choice of H, and $\widehat{T}$ is injective, since $\widehat{T}(g, h) = 0$ implies $Tg = -h \in R(T) \cap H = \{0\}$, and $T|_G$ is injective. Hence we find $\varepsilon > 0$ such that, for $\tilde{S} \in \mathcal{L}(G \times H, Y)$, $\|\tilde{S} - \widehat{T}\| < \varepsilon$ implies that $\tilde{S} : G \times H \to Y$ is an isomorphism. end Tue 21.05.19

Now let $S \in \mathcal{L}(X, Y)$ with $\|S - T\| < \varepsilon$. Then $\|\widehat{S} - \widehat{T}\| < \varepsilon$, and $\widehat{S} : G \times H \to Y$ is an isomorphism. We now show

(i) $\alpha(S) \le \alpha(T) < \infty$,

(ii) $\beta(S) \le \beta(T) < \infty$,

(iii) $\operatorname{ind}(S) = \operatorname{ind}(T)$.

(i): If $g \in N(S) \cap G$ then $\widehat{S}(g,0) = 0$, and $g = 0$. Hence $N(S) \cap G = \{0\}$ and

$$\alpha(S) = \dim N(S) \le \operatorname{codim} G = \dim N(T) = \alpha(T) < \infty.$$

(ii): $G \times \{0\}$ is closed in $G \times H$, so $S(G) = \widehat{S}(G \times \{0\})$ is closed in Y. Since $\widehat{S}$ is surjective we have $Y = S(G) + H$. If $y = S(g) \in S(G) \cap H$ then $\widehat{S}(g, -y) = S(g) - S(g) = 0$ and $g = 0$, $y = 0$, since $\widehat{S}$ is injective. Hence $S(G) \oplus H = Y = T(G) \oplus H$ and

$$\beta(S) = \operatorname{codim} R(S) \le \operatorname{codim} S(G) = \operatorname{codim} T(G) = \operatorname{codim} R(T) = \beta(T) < \infty.$$

(iii): By what we have already shown we know that $N(S) \oplus G$ is a closed subspace of X of finite codimension. We thus find a subspace W of X with $\dim W < \infty$ such that $X = W \oplus N(S) \oplus G$. We then have

$$\alpha(T) = \dim N(T) = \dim N(S) \oplus \dim W = \alpha(S) + \dim W.$$

On the other hand, we have

$$R(S) = S(X) = S(G \oplus W) = S(G) \oplus S(W)$$

since both subspaces on the right hand side are closed and their intersection is trivial (by injectivity of $S|_{G\oplus W}$). Hence

$$\beta(T) = \operatorname{codim} T(X) \overset{\text{(ii)}}{=} \operatorname{codim} S(G) = \operatorname{codim} S(X) + \dim S(W) = \beta(S) + \dim W,$$

and we conclude that

$$\operatorname{ind} T = \alpha(T) - \beta(T) = \alpha(S) + \dim W - (\beta(S) + \dim W) = \operatorname{ind} S,$$

which ends the proof. □

11.4. Compact Operators: Recall (cf. FA) that a linear operator $T : X \to Y$ is called *compact* if $\overline{T(B_X)}$ is compact in Y where $B_X := \{x \in X : \|x\| \le 1\}$ denotes the closed unit ball in X. The set of all compact linear operators from X to Y is denoted by $\mathcal{K}(X,Y)$. We recall the following properties ($\to$ FA):

- Any compact linear operator $T : X \to Y$ is bounded, i.e. $\mathcal{K}(X,Y) \subseteq \mathcal{L}(X,Y)$.
- $\dim T(X) < \infty \Longrightarrow T$ is compact.

- I_X is compact $\iff \dim X < \infty$.
- $\mathcal{K}(X,Y)$ is a closed linear subspace of $\mathcal{L}(X,Y)$.
- If W, Z are Banach spaces and $S \in \mathcal{L}(W,X)$, $R \in \mathcal{L}(X,Y)$ and $T \in \mathcal{K}(X,Y)$ then $RT \in \mathcal{K}(X,Z)$ and $TS \in \mathcal{K}(W,Y)$ (*ideal property*).
- **Schauder's Theorem:** For any $T \in \mathcal{L}(X,Y)$: $T \in \mathcal{K}(X;Y) \iff T' \in \mathcal{K}(Y',X')$.

11.5. Proposition: Let $K \in \mathcal{K}(X)$. Then $I - K \in \Phi(X)$ and $\operatorname{ind}(I-K) = 0$.

Proof. On $N := N(I-K)$ we have $I_N = K|_N \in \mathcal{K}(N)$ and thus $\dim N < \infty$. We let $T := I - K$ and show that $R(T)$ is closed. To this end we factorize $T = S \circ q$ where $q : X \to X/N$ is the quotient map and $S : X/N \to X$. Then $R(T) = R(S)$ and S is injective. We claim that there exists $\eta > 0$ such that $\|Sw\| \geq \eta\|w\|$ for all $w \in X/N$. From this, closedness of $R(S) = R(T)$ follows.

Assume that such an η does not exist. Then we find a sequence (w_n) such that $\|w_n\| = 1$ and $Sw_n \to 0$, and a sequence (x_n) in X such that $q(x_n) = w_n$, $\|x_n\| \leq 2$. Then $Tx_n = Sw_n \to 0$ and $d(x_n, N) = \|q(x_n)\| = 1$.

Since K is compact we have $Kx_{k(n)} \to y \in X$ for a subsequence, which implies $x_{k(n)} = Tx_{k(n)} + Kx_{k(n)} \to y$ and $Sw_{k(n)} = Tx_{k(n)} \to Ty$. So $Ty = 0$ and $y \in N$. However, $d(y,N) = 1$, a contradiction.

By Schauder's Theorem (cf. above) $K' \in \mathcal{K}(X')$, hence $\dim N((I-K)') < \infty$. Since $R(I-K)$ is closed we obtain

$$\begin{aligned}\operatorname{codim} R(I-K) &= \dim X/R(I-K) = \dim\left(X/R(I-K)\right)' \\ &= \dim\{\phi \in X' : \phi|_{R(I-K)} = 0\} = \dim\left(N(I-K)'\right) < \infty.\end{aligned}$$

It remains to show $\operatorname{ind}(I-K) = 0$. We consider $\gamma : [0,1] \to \mathbb{Z}$, $t \mapsto \operatorname{ind}(I - tK) = 0$ (observe that $tK \in \mathcal{K}(X)$ and hence $I - tK \in \Phi(X)$ for any $t \in \mathbb{R}$). By 11.3, γ is continuous, hence constant and $\operatorname{ind}(I-K) = \gamma(1) = \gamma(0) = \operatorname{ind} I = 0$. □

11.6. Corollary: Let $K \in \mathcal{K}(X)$ and $\dim X = \infty$. Then $0 \in \sigma(K)$ and for $\lambda \in \sigma(K) \setminus \{0\}$ one has that $\lambda - K$ is a Fredholm operator of index 0. For $\lambda \in \mathbb{C} \setminus \{0\}$ one has in particular

$$\lambda - K \text{ is injective} \quad \iff \quad \lambda - K \text{ is surjective.}$$

and, more precisely, the following **Fredholm alternative**:

Either $(\lambda - K)x = 0$ has just the trivial solution; in this case $(\lambda - K)x = y$ has a unique solution for any $y \in X$,

or $(\lambda - K)x = 0$ has exactly $n \in \mathbb{N}$ linearly independent solutions and also the dual equation $(\lambda - K')\phi = 0$ has exactly n linearly independent solutions; in this case $(\lambda - K)x = y$ has solutions if and only if $\phi(y) = 0$ for any $\phi \in N(\lambda - K')$.

11.7. Corollary: Let $T \in \mathcal{L}(X,Y)$ be an isomorphism from X to Y and $K \in \mathcal{K}(X,Y)$. Then $T-K \in \Phi(X,Y)$ and $\operatorname{ind}(T-K)=0$.

Proof. We have $T^{-1}K \in \mathcal{K}(X)$, hence by Proposition 11.5, $T^{-1}(T-K) = I_X - T^{-1}K \in \Phi(X)$ and $\operatorname{ind} T^{-1}(T-K) = 0$. Since T^{-1} is an isomorphism from Y to X we obtain the assertion. □

The assertion in 11.7 is the keystone for the following powerful perturbation property of Fredholm operators.

11.8. Theorem: Let $T \in \Phi(X,Y)$ and $K \in \mathcal{K}(X,Y)$. Then $T+K \in \Phi(X,Y)$ and $\operatorname{ind}(T+K) = \operatorname{ind} T$.

end Thu 23.05.19

Proof. We take up the construction and notation of the proof of 11.3 and write

$$X = N(T) \oplus G,\ Y = R(T) \oplus H,\ \dim H < \infty.$$

where G is a closed subspace of X. We let $S := T-K$ and consider the operators $\widehat{T}, \widehat{S} : G \times H \to Y$. Again, $\widehat{T}$ is an isomorphism, and we have $\widehat{S} = \widehat{T} - \widetilde{K}$ where $\widetilde{K}(g,h) := Kg$. The operator $\widetilde{K}$ is compact, thus by 11.7 we have $\widehat{S} \in \Phi(G \times H, Y)$ and $\operatorname{ind} \widehat{S} = 0$, i.e. $\alpha(\widehat{S}) = \beta(\widehat{S})$. Clearly,

$$(N(S) \cap G) \times \{0\} = \{(g,0) \in G \times H : Sg = 0\} \subseteq N(\widehat{S}),$$

which implies

$$\alpha(S) = \dim N(S) \le \operatorname{codim} G + \alpha(\widehat{S}) = \alpha(T) + \alpha(\widehat{S}) < \infty.$$

Moreover, we have $\widehat{S}(G \times H) = S(G) + H$, and $S(G)$ has finite codimension and is thus closed by 11.2. We obtain

$$\beta(S) = \operatorname{codim} S(X) \le \operatorname{codim} S(G) \le \operatorname{codim} \widehat{S}(G \times H) + \dim H = \beta(\widehat{S}) + \beta(T) < \infty,$$

and $S(X)$ is closed by 11.2.

For the calculation of the index we have to look closer. We let $G_0 := N(S) \cap G$, find a complement G_1 of G_0 in G, a complement W_0 of G_0 in $N(S)$, and a complement W_1 of $N(S) + G = W_0 \oplus G_0 \oplus G_1$ in X. Then we have

$$\begin{aligned}
\alpha(T) &= \dim W_1 + \dim W_0,\\
\beta(T) &= \dim H,\\
\alpha(S) &= \dim W_0 + \dim G_0,\\
\beta(S) &= \operatorname{codim} S(X) = \operatorname{codim} S(G_1) - \dim W_1,
\end{aligned}$$

and

$$\begin{aligned}\beta(\widehat{S}) &= \operatorname{codim}(S(G)+H) = \operatorname{codim} S(G) - \dim H + \dim(S(G)\cap H)\\ &= \operatorname{codim} S(G) - \dim H + \dim(S(G_1)\cap H).\end{aligned}$$

Now

$$\dim(S(G_1)\cap H) = \dim\{g\in G_1 : Sg\in H\} = \dim \underbrace{\{(g,h)\in G_1\times H : Sg=-h\}}_{=:V},$$

and V is a complement of $G_0\times\{0\}$ in $N(\widehat{S})$. We thus obtain

$$\alpha(\widehat{S}) = \dim N(\widehat{S}) = \dim G_0 + \dim V = \dim G_0 + \dim(S(G_1)\cap H).$$

By $\alpha(\widehat{S}) = \beta(\widehat{S})$ we obtain

$$\dim G_0 = \operatorname{codim} S(G_1) - \dim H.$$

Putting everything together, we have

$$\begin{aligned}\operatorname{ind} S &= \alpha(S) - \beta(S)\\ &= \dim W_0 + \dim G_0 - \operatorname{codim} S(G_1) + \dim W_1\\ &= \alpha(T) - \operatorname{codim} S(G_1) + \dim G_0\\ &= \alpha(T) - \dim H = \alpha(T) - \beta(T) = \operatorname{ind} T,\end{aligned}$$

which ends the proof. □

11.9. Theorem: Let $K\in\mathcal{K}(X)$ and $\lambda\in\sigma(K)\setminus\{0\}$, and denote, for $k\in\mathbb{N}$, $N_k := N((\lambda-K)^k)$ and $R_k := R((\lambda-K)^k)$. Then one has

(a) For each $k\in\mathbb{N}$, N_k and R_k are closed and $\dim N_k = \operatorname{codim} R_k < \infty$.
(b) There is a minimal $p\in\mathbb{N}$ such that $N_p = N_{p+1}$.
(c) $N_{p+k} = N_p$ and $R_{p+k} = R_p$ for each $k\in\mathbb{N}$.
(d) $X = N_p\oplus R_p$, $(\lambda-K)N_p\subseteq N_p$, $((\lambda-K)|_{N_p})^p = 0$ and $\lambda-K : R_p\to R_p$ is an isomorphism.

In particular, $\sigma(K)\setminus\{0\}\subseteq\sigma_p(K)$. Moreover, $\sigma(K)\setminus\{0\}$ is finite or $\sigma(K)\setminus\{0\} = \{\lambda_n : n\in\mathbb{N}\}$ where $\lambda_n\to 0$.

Assertion (d) means that $X = N_p\oplus R_p$ where N_p and R_p are invariant under the operator K. Thus K corresponds to a matrix operator $\begin{pmatrix} K|_{N_p} & 0\\ 0 & K|_{R_p}\end{pmatrix} : N_p\times R_p\to N_p\times R_p$ where $\sigma(K|_{N_p}) = \{\lambda\}$ and $\sigma(K|_{R_p}) = \sigma(K)\setminus\{\lambda\}$. The space N_p is finite-dimensional and $K|_{N_p}$ corresponds to a matrix with single eigenvalue λ.

Proof. (a) Writing $(\lambda - K)^k = \lambda^k(I - KM_k)$ where $M_k \in \mathcal{L}(X)$ we see by 11.5 that $(\lambda - K)^k \in \Phi(X)$ and $\operatorname{ind}(\lambda - K)^k = 0$.

(b) If not, we find a sequence (x_n) in X such that, for each n, $x_n \in N_{n+1} \setminus N_n$ and $1 = d(x_n, N) \leq \|x_n\| \leq 2$. For $n > m$ we then have

$$\|Kx_n - Kx_m\| = \|\lambda x_n - \underbrace{((\lambda - K)x_n + Kx_m)}_{=:y}\|$$

and

$$(\lambda - K)^n y = (\lambda - K)^{n+1}x_n + K(\lambda - K)^n x_m = 0,$$

i.e. $y \in N_n$. Hence

$$\|Kx_n - Kx_m\| = \|\lambda x_n - y\| \geq |\lambda| d(x_n, N_n) = |\lambda| > 0,$$

in contradiction with $K \in \mathcal{K}(X)$.

(c) If $(\lambda - K)^{p+2}x = 0$ then $(\lambda - K)x \in N_{p+1} = N_p$ and $(\lambda - K)^{p+1}x = 0$, i.e. $x \in N_{p+1} = N_p$. We have shown $N_{p+2} = N_p$. Now we iterate and use (a).

(d) For each k, $\lambda - K : R_k \to R_{k+1}$ is surjective. Hence $\lambda - K : R_p \to R_p$ is surjective. But by 11.5, $(\lambda - K)|_{R_p} \in \Phi(R_p)$ has index 0, so $\lambda \in \rho(K|_{R_p})$. On the other hand, $\lambda - K : N_{k+1} \to N_k$ for each k, hence $\lambda - K : N_p \to N_p$ and $((\lambda - K)|_{N_p})^p = 0$. Since $\dim N_p < \infty$ we obtain from linear algebra that $\sigma(K|_{N_p}) = \{\lambda\}$.

In particular, there exists $\varepsilon > 0$ such that $B(\lambda, \varepsilon) \setminus \{\lambda\} \subset \rho(A)$. This implies that the only possible accumulation point of $\sigma(K) \setminus \{0\}$ is 0. □

Example: Let $X = C[0,1]$ and define $V \in \mathcal{L}(X)$ by $Vf(t) := \int_0^t f(s)\,ds$. Then $V : X \to X$ is compact, but $\sigma_p(V) = \emptyset$, since $\lambda f = Vf$ implies $f(0) = 0$ and $\lambda f' = f$, hence $f = 0$. By 11.6, $\sigma(V) = \{0\}$.

Via $(V^n f)(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)\,ds$ it is also possible to show $\|V^n\| \leq \frac{1}{n!}$ which implies $r(V) = 0$ for the spectral radius of V.

We want to have a corresponding result for unbounded operators, and this can be achieved via resolvents.

11.10. Lemma: Let A be a closed operator in X with non-empty resolvent set. The following are equivalent:

(i) $I : [D(A)] \to X$ is compact,

(ii) there exists $\lambda_0 \in \rho(A)$ such that $R(\lambda_0, A) \in \mathcal{K}(X)$,

(iii) $R(\lambda, A) \in \mathcal{K}(X)$ for all $\lambda \in \rho(A)$.

In this case we say that A *has compact resolvents*.

Proof. (i) $\Rightarrow$ (ii): We choose $\lambda_0 \in \rho(A)$ and factorize $R(\lambda_0, A) = R(\lambda_0, A) \circ I$ where $R(\lambda_0, A) \in \mathcal{L}(X, [D(A)])$ and $I \in \mathcal{K}([D(A)], X)$.

(ii) $\Rightarrow$ (iii): Use the resolvent equation.

(iii) $\Rightarrow$ (i): Choose $\lambda_0 \in \rho(A)$ and factorize $I = R(\lambda_0, A)(\lambda_0 - A)$ where $\lambda_0 - A \in \mathcal{L}([D(A)], X)$ and $R(\lambda_0, A) \in \mathcal{K}(X)$. □

end Tue 28.05.19

11.11. Lemma: Let A be a closed operator in X, $\lambda_0 \in \rho(A)$ and $z \in \mathbb{C} \setminus \{\lambda_0\}$. For all $k \in \mathbb{N}$ we then have

$$N\big((z-A)^k\big) = N\big(\big((\lambda_0-z)^{-1} - R(\lambda_0, A)\big)^k\big), \quad R\big((z-A)^k\big) = R\big(\big((\lambda_0-z)^{-1} - R(\lambda_0, A)\big)^k\big),$$

In particular, $z - A \in \Phi([D(A)], X)$ if and only if $(\lambda_0 - z)^{-1} - R(\lambda_0, A) \in \Phi(X)$. In this case, one has $\operatorname{ind}(z - A) = \operatorname{ind}((\lambda_0 - z)^{-1} - R(\lambda_0, A))$.

Proof. is an exercise. □

Combination of 11.9, 11.10, and 11.11 yields the following version of Theorem 11.9 for unbounded operators.

11.12. Theorem: Let A be a closed linear operator in X with $\rho(A) \neq \emptyset$ and compact resolvents. Let $z \in \sigma(A)$ and denote, for $k \in \mathbb{N}$, $N_k := N((z-A)^k)$ and $R_k := R((z-A)^k)$. Then one has

(a) For each $k \in \mathbb{N}$, N_k and R_k are closed and $\dim N_k = \operatorname{codim} R_k < \infty$.

(b) There is a minimal $p \in \mathbb{N}$ such that $N_p = N_{p+1}$.

(c) $N_{p+k} = N_p$ and $R_{p+k} = R_p$ for each $k \in \mathbb{N}$.

(d) $X = N_p \oplus R_p$, $(z-A)N_p \subset N_p$, $((z-A)|_{N_p})^p = 0$ and $A|_{R_p}$ with $D(A|_{R_p}) = D(A) \cap R_p$ is a closed linear operator in R_p with $z \in \rho(A|_{R_p})$.

In particular, $\sigma(A) = \sigma_p(A)$. Moreover, $\sigma(A)$ is finite or $\sigma(A) = \{z_n : n \in \mathbb{N}\}$ where $|z_n| \to \infty$.

Observe that, for a fixed $\lambda_0 \in \rho(A)$, the operator $(\lambda_0 - z)^{-1} - R(\lambda_0, A) : R_p \to R_p$ is by 11.11 and 11.9 an isomorphism and that $R(\lambda_0, A)(R_p) = R_p \cap D(A) = D(A|_{R_p})$.

Example: Let $X = \{f \in C[0,1] : f(0) = f(1)\}$ and $A = \frac{d}{dx}$ with $D(A) = \{f \in X \cap C^1[0,1] : f' \in X\}$. Cleraly, A is closed and, by Arzelá-Ascoli, the embedding $I : [D(A)] \to X$ is compact. We show $\lambda \in \rho(A)$ for $\lambda \notin 2\pi i\mathbb{Z}$: The equation $(\lambda - A)f = g$ means $\lambda f - f' = g$, and the general solution is given by

$$f(x) = ce^{\lambda x} - e^{\lambda x} \int_0^x e^{-\lambda y} g(y)\, dy, \quad x \in [0,1],$$

where $c \in \mathbb{C}$. The condition $f(0) = f(1)$ leads to

$$c = c(g) = \frac{e^\lambda}{e^\lambda - 1} \int_0^1 e^{-\lambda y} g(y)\, dy,$$

observe that $e^\lambda \neq 1$ by $\lambda \notin 2\pi i\mathbb{Z}$. Hence $\lambda - A$ is bijective and $\lambda \in \rho(A)$. For $\lambda \in 2\pi i\mathbb{Z}$ we see that $f(x) = e^{\lambda x}$, $x \in [0,1]$, satisfies $f \in D(A)$ and $(\lambda - A)f = 0$ and spans $N(\lambda - A)$. The operator A has compact resolvents and $\sigma(A) = \sigma_p(A) = 2\pi i\mathbb{Z}$. For $z \in \sigma(A)$ we have $\alpha(z - A) = 1 = \beta(z - A)$.

Regarding Fredholm operators as generalizations of isomorphisms leads to the following notion.

11.13. Definition: Let A be a closed linear operator in X. We define the *essential spectrum* of A by

$$\sigma_{\text{ess}}(A) := \{\lambda \in \mathbb{C} : \lambda - A \notin \Phi([D(A)], X)\}.$$

Clearly, $\sigma_{\text{ess}}(A) \subseteq \sigma(A)$. Any $\lambda \in \mathbb{C} \setminus \sigma_{\text{ess}}(A)$ is called a *Fredholm point for A*.

Example: If $\dim X < \infty$ then $\sigma_{\text{ess}}(A) = \emptyset$ for any $A \in \mathcal{L}(X)$. If $K \in \mathcal{K}(X)$ and $\dim X = \infty$ then $\sigma_{\text{ess}}(K) = \{0\}$ ($R(K)$ is either finite dimensional or not closed). If A is a closed operator in X with non-empty resolvent set and compact resolvents then $\sigma_{\text{ess}}(A) = \emptyset$.

Remarks: (a) Observe that $\sigma_c(A) \subseteq \sigma_{\text{ess}}(A)$.

(b) By 11.5, $\sigma_{\text{ess}}(A)$ is closed, $\mathbb{C} \setminus \sigma_{\text{ess}}(A)$ is open, and the map $\mathbb{C} \setminus \sigma_{\text{ess}}(A) \to \mathbb{Z}$, $\lambda \mapsto \operatorname{ind}(\lambda - A)$ is continuous.

(c) By 11.11 we have: If $\lambda_0 \in \rho(A)$ then

$$\sigma_{\text{ess}}(R(\lambda_0, A)) \setminus \{0\} = \Big\{ \frac{1}{\lambda_0 - z} : z \in \sigma_{\text{ess}}(A) \Big\}.$$

Example: Let $X = \{f \in C[0,\infty) : \lim_{x\to\infty} f(x) = 0\}$ with sup-norm $\|\cdot\|_\infty$, and $A = \frac{d}{dx}$, $D(A) = \{f \in X \cap C^1[0,\infty) : f' \in X\}$. For $\lambda \in \mathbb{C}$ and $f \in D(A)$ we have $(\lambda - A)f = 0$ if and only if $f' = \lambda f$, i.e. $f = ce^{\lambda(\cdot)}$. We thus obtain $\sigma_p(A) = \{\operatorname{Re}\lambda < 0\}$.

For $\operatorname{Re}\lambda > 0$ and $g \in X$ the unique solution $f \in D(A)$ of $(\lambda - A)f = g$ is given by

$$f(x) = ce^{\lambda x} - e^{\lambda x} \int_0^x e^{-\lambda t} g(t)\, dt,$$

where $c = c(g)$ has to be chosen such that $\lim_{x\to\infty} f(x) = 0$, i.e.

$$c(g) = \int_0^\infty e^{-\lambda t} g(t)\, dt,$$

so that

$$\big((\lambda - A)^{-1}g\big)(x) = \int_x^\infty e^{\lambda(x-t)} g(t)\,dt = \int_0^\infty e^{-\lambda s} g(x+s)\,ds,$$

observe that, by dominated convergence, the integral on the right hand side tends to 0 for $x \to \infty$. We obtain $\{\operatorname{Re}\lambda > 0\} \subseteq \rho(A)$. For $\operatorname{Re}\lambda < 0$ and $g \in X$ the unique solution of $(\lambda - A)f = g$, $f(0) = 0$ is given by

$$f(x) = \int_0^x e^{\lambda(x-t)} g(t)\,dt.$$

Observe that $f \in D(A)$: extending g by 0 to a function on $\mathbb{R}$ we have

$$|f(x)| \le \int_0^\infty e^{(\operatorname{Re}\lambda)y} |g(x-y)|\,dy$$

which tends to 0 as $x \to \infty$ by dominated convergence. This means that $R(\lambda - A) = X$, $\beta(\lambda - A) = 0$, $\lambda - A \in \Phi(X)$ and $\operatorname{ind}(\lambda - A) = 1$ for any $\operatorname{Re}\lambda < 0$. Continuity of the index then yields that $\sigma_{\mathrm{ess}}(A) = i\mathbb{R}$. end Tue 04.06.19

11.14. Lemma: Let $T \in \Phi(X, Y)$. Then $T' \in \Phi(Y', X')$ and

$$\alpha(T') = \beta(T), \quad \beta(T') = \alpha(T), \quad \operatorname{ind} T' = -\operatorname{ind} T.$$

Proof. is an exercise. □

From this lemma we obtain via spectral mapping:

11.15. Corollary: Let A be a closed and densely defined linear operator in X with $\rho(A) \neq \emptyset$. Then $\sigma_{\mathrm{ess}}(A') = \sigma_{\mathrm{ess}}(A)$ and, for $z \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(A)$,

$$\alpha(z - A') = \beta(z - A), \quad \beta(z - A') = \alpha(z - A), \quad \operatorname{ind}(z - A') = -\operatorname{ind}(z - A).$$

11.16. Proposition: Let A be a closed operator in X and $K \in \mathcal{K}([D(A)], X)$. Then the Banach spaces $[D(A)]$ and $[D(A+K)]$ are isomorphic and one has

$$\sigma_{\mathrm{ess}}(A + K) = \sigma_{\mathrm{ess}}(A).$$

Proof. Clearly, $D(A+K) = D(A)$ as sets and

$$\|x\|_{A+K} = \|x\| + \|(A+K)x\| \le \|x\| + \|Ax\| + \|K\|\|x\|_A \le \big(1 + \|K\|\big)\|x\|_A.$$

Now assume that there is no constant $C > 0$ with $\|x\|_A \le C\|x\|_{A+K}$. Then we find a sequence (x_n) in $D(A)$ with $\|x_n\|_A = 1$ and $\|x_n\|_{A+K} \to 0$. Since K is compact $[D(A)] \to X$ we find a convergent subsequence of (Kx_n) and may assume $Kx_n \to y$ in X. By $\|x_n\|_{A+K} \to 0$ we have $\|x_n\| \to 0$ and $\|(A+K)x_n\| \to 0$. Hence

$$Ax_n = (A+K)x_n - Kx_n \to 0 - y = y \quad \text{in } X.$$

Since A is closed we infer $y = 0$, hence also $Ax_n \to 0$ in X, i.e. $\|x_n\|_A \to 0$, a contradiction. The assertion on the essential spectrum follows from 11.8. □

We give somehow typical examples.

11.17. Example: Let $X = \{f \in C[0,\infty) : \lim_{x\to\infty} f(x) = 0\}$ with sup-norm $\|\cdot\|_\infty$, and $A = \frac{d}{dx}$, $D(A) = \{f \in X \cap C^1[0,\infty) : f' \in X\}$. We know that $\sigma_{\text{ess}}(A) = i\mathbb{R}$.

Let $m \in C[0,\infty)$ such that $m(t) = 0$ for $t \geq a$ where $a > 0$. Then $f \mapsto mf$ is compact from $[D(A)]$ to X (by Arzelá-Ascoli). Hence, if we define $Bf := f' + mf$ for $f \in D(B) := D(A)$, then $\sigma_{\text{ess}}(B) = i\mathbb{R}$.

By an approximation argument, the same holds for $m \in X$.

11.18. Example: In $X = L^2(\mathbb{R}^d)$ we consider $A = -\Delta$, defined as in 10.20 on $D(A) = W^{2,2}(\mathbb{R}^d) = H^2(\mathbb{R}^d)$ by

$$-\Delta f = \sum_{j=1}^{d} \partial_j^2 f = \mathscr{F}^{-1}(\xi \mapsto 4\pi^2|\xi|^2 \hat{f}(\xi)).$$

We know that A is closed, the graph norm is equivalent to $\|\cdot\|_{H^2}$, and $\sigma(A) = [0,\infty)$: For $\lambda \in \mathbb{C} \setminus [0,\infty)$ we have $\lambda \in \rho(A)$ and

$$R(\lambda, A)f = (\lambda + \Delta)^{-1} f = \mathscr{F}^{-1}(\xi \mapsto \underbrace{(\lambda - 4\pi^2|\xi|^2)^{-1}}_{\in L^\infty} \hat{f}(\xi)), \quad f \in L^2(\mathbb{R}^d).$$

Now we show that $\sigma_{\text{ess}}(A) = [0,\infty)$: For $\lambda \geq 0$ we have $N(\lambda - A) = \{0\}$ since $(\lambda - A)f = 0$ implies $(\lambda - 4\pi^2|\xi|^2)\hat{f}(\xi) = 0$ for a.e. $\xi \in \mathbb{R}^d$ and then $\hat{f} = 0$ a.e. and $f = 0$ a.e. as $\{\xi : |\xi| = \sqrt{\lambda}\}$ is a null-set. Moreover,

$$R(\lambda - A) = \mathscr{F}^{-1}((\lambda - 4\pi^2|\cdot|^2)\mathscr{F}H^2(\mathbb{R}^d)) = \mathscr{F}^{-1}((\lambda - 4\pi^2|\cdot|^2)(1 + 4\pi^2|\cdot|^2)^{-1}L^2(\mathbb{R}^d))$$

contains the dense set

$$\mathscr{F}^{-1}(\{f \in L^2 : \exists \varepsilon > 0 : f(\xi) = 0 \text{ if } |\xi| \geq \tfrac{1}{\varepsilon} \text{ or } ||\xi| - \sqrt{\lambda}| < \varepsilon\})).$$

So $R(\lambda - A)$ is dense in $L^2(\mathbb{R}^d)$ but not closed in $L^2(\mathbb{R}^d)$ (since $(\lambda - A)^{-1}$ is unbounded). Hence $\sigma(A) = \sigma_{\text{ess}}(A) = [0,\infty)$.

Now we let $m_j \in L^\infty(\mathbb{R}^d)$ for $j = 0, 1, \ldots, d$ with $m_j(x) = 0$ for $|x| \geq a$ where $a > 0$ is fixed. We define $Bf := \sum_{j=1}^d m_j \partial_j f + m_0 f$ for $f \in H^2(\mathbb{R}^d)$. We show that $B : [D(A)] \to L^2(\mathbb{R}^d)$ is compact, i.e. $B \in \mathcal{K}(H^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$.

To this end we choose $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi = 1$ on $B(0, a+1)$ and $\operatorname{supp}\chi \subseteq B(0, a+2)$ and observe $Bf = \sum_{j=1}^d m_j \chi \partial_j f + m_0 \chi f$. The operators $f \mapsto \chi \partial_j f$, $f \mapsto \chi f$ are bounded $H^2(\mathbb{R}^d) \to H_0^1(B(0, a+2))$, the embedding $H_0^1(B(0, a+2)) \to L^2(\mathbb{R}^d)$ is compact, and the multiplication operators $f \mapsto m_j f$ are bounded on $L^2(\mathbb{R}^d)$.

We conclude that $B : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is compact as claimed. By 11.16 we thus have $\sigma_{\text{ess}}(-\Delta + B) = \sigma_{\text{ess}}(-\Delta) = [0,\infty)$.

12 Operators in Hilbert space

When nothing else is said, H denotes a complex Hilbert space with inner product $(\cdot|\cdot)$.

Recall the following properties of an inner (or *scalar*) product:

- $(\cdot|\cdot)$ is linear in the first component,
- $(y|x) = \overline{(x|y)}$ for all $x, y \in H$,
- these two properties imply that $(\cdot|\cdot)$ is *antilinear* in the second component, i.e. $(x|\alpha y + z) = \overline{\alpha}(x|y) + (x|z)$ for al $x, y, z \in H$, $\alpha \in \mathbb{C}$.
- $(x|x) \geq 0$ and $(x|x) = 0 \iff x = 0$ for all $x \in H$.

These properties imply that $x \mapsto \|x\| := \sqrt{(x|x)}$ defines a norm on H and that the *Cauchy-Schwarz inequality*

$$|(x|y)| \leq \|x\|\|y\|$$

holds for all $x, y \in H$.

A space H equipped with such an inner product $(\cdot|\cdot)$ is called a *Hilbert space* if it is complete for the norm $\|\dot{}\|$ associated with the inner product.

Let $H' := \mathcal{L}(H, \mathbb{C})$ denote the dual space of H. Then the map

$$J_H : H \mapsto H', \quad y \mapsto (\cdot|y)$$

is bijective, isometric, and antilinear. Let H^* denote the anti-dual space of H, i.e. the linear space of all continuous antilinear functionals $H \to \mathbb{C}$. Then the map

$$\overline{J}_H : H \mapsto H^*, \quad x \mapsto (x|\cdot)$$

is bijective, isometric, and linear. end Thu 06.06.19

12.1. Definition: Let H_1, H_2 be Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$. The *adjoint operator* $T^* \in \mathcal{L}(H_2, H_1)$ of T is defined by

$$(x|T^*y)_{H_1} = (Tx|y)_{H_2} \quad \text{for all } x \in H_1,\, y \in H_2,$$

i.e. $T^* = J_{H_1}^{-1} T' J_{H_2}$ where $T' \in \mathcal{L}(H_2', H_1')$ denotes the dual operator of T. We clearly have

$$\|T^*\|_{\mathcal{L}(H_2,H_1)} = \|T\|_{\mathcal{L}(H_1,H_2)}.$$

In the same way we can define the adjoint operator A^* as an operator from H_2 to H_1 for a densely defined operator $A : H_1 \supseteq D(A) \to H_2$.

An operator $S \in \mathcal{L}(H)$ is called *self-adjoint* if $S^* = S$, *normal* if $SS^* = S^*S$, and *unitary* if $SS^* = S^*S = I$.

Rules: $(T^*)^* = T$, $(S+T)^* = S^* + T^*$, $(\alpha T)^* = \overline{\alpha} T^*$, $(ST)^* = T^* S^*$.

Remark: For $S \in \mathcal{L}(H)$ we have

$$\begin{aligned} S \text{ self-adjoint} &\iff \forall x, y \in H : (Sx|y) = (x|Sy) \\ S \text{ normal} &\iff \forall x, y \in H : (Sx|Sy) = (S^*x|S^*y) \\ S \text{ unitary} &\iff S \text{ is bijective and isometric.} \end{aligned}$$

12.2. Lemma: Let $S \in \mathcal{L}(H)$.

(a) $\|S^*S\| = \|SS^*\| = \|S\|^2$ and S^*S, SS^* are self-adjoint.

(b) If S is normal, then

$$\|S\| = r(S) = \max\{|\lambda| : \lambda \in \sigma(S)\}.$$

Proof. (a) We have

$$\|Sx\|^2 = |(Sx|Sx)| = |(x|S^*Sx)| \le \|x\|\|S^*Sx\|,$$

hence

$$\|S\|^2 \le \|S^*S\| \le \|S^*\|\|S\| = \|S\|^2.$$

(b) We calculate $r(S)$. Using (a) repeatedly and normality of S once we have

$$\|S^2\|^2 = \|S^2(S^2)^*\| = \|(SS^*)^2\| = \|(SS^*)(SS^*)^*\| = \|SS^*\|^2 = \|S\|^4.$$

Hence $\|S^2\| = \|S\|^2$. Iteration yields $\|S^{2^k}\| = \|S\|^{2^k}$ for any $k \in \mathbb{N}$, from which $r(S) = \|S\|$ follows. □

12.3. Definition und Lemma: Let $T \in \mathcal{L}(H)$ and

$$W(T) := \{(Tx|x) : \|x\| = 1\}$$

be the *numerical range* of T. Then $\sigma(T) \subseteq \overline{W(T)}$.

Proof. Let $\lambda \notin \overline{W(T)}$. Then $d := d(\lambda, \overline{W(T)}) > 0$ and for $\|x\| = 1$:

$$d \le |\lambda - (Tx|x)| = |((\lambda - T)x|x)| \le \|(\lambda - T)x\| \cdot \|x\| = \|(\lambda - T)x\|.$$

Hence $\lambda - T$ is injective, $(\lambda - T)^{-1} : R(\lambda - T) \to H$ is bounded, and $R(\lambda - T)$ is closed.[3] Now $R(\lambda - T)$ is dense if and only if $(\lambda - T)^* = \overline{\lambda} - T^*$ is injective. Since $W(T^*) = \{\overline{\mu} : \mu \in W(T)\}$, we have $d(\overline{\lambda}, \overline{W(T^*)}) = d(\lambda, W(T)) = d > 0$. The previous argument shows that $\overline{\lambda} - T^*$ is injective. □

[3] see Exercise 29, FA

12.4. Corollary: If $S \in \mathcal{L}(H)$ is self-adjoint then $W(S) \subseteq \mathbb{R}$ and

$$\sigma(S) \subseteq [m, M] \subseteq [-\|S\|, \|S\|],$$

where $m := \inf\{(Sx|x) : \|x\| = 1\}$ and $M := \sup\{(Sx|x) : \|x\| = 1\}$. Moreover, $m, M \in \sigma(S)$ and $M = \|S\|$ or $m = -\|S\|$.

Proof. For $x \in H$ with $\|x\| = 1$ we have

$$(Sx|x) = (x|Sx) = \overline{(Sx|x)},$$

i.e. $(Sx|x) \in \mathbb{R}$, and

$$|(Sx|x)| \leq \|Sx\|\|x\| \leq \|S\|,$$

which implies $[m, M] \subseteq [-\|S\|, \|S\|]$. The inclusion $\sigma(S) \subseteq [m, M]$ follows from 12.3. By 12.2 we have $r(S) = \|S\|$. By $W(S - m) = W(S) - m$, $\sigma(S - m) = \sigma(S) - m$, $r(S - m) = \|S - m\|$ we infer $\|S - m\| = M - m \in \sigma(S - m)$, i.e. $M \in \sigma(S)$. In the same way, $W(S - M) = W(S) - M$ etc and $-\|S - M\| = m - M \in \sigma(S - M)$, i.e. $m \in \sigma(S)$. Then $r(S) = \|S\|$ implies $M = \|S\|$ or $m = -\|S\|$. $\square$

The following is a first spectral theorem for bounded self-adjoint operators and establishes a *functional calculus* for functions that are continuous on the spectrum. We write $p_j(\lambda) = \lambda^j$ for $j \in \mathbb{N}_0$, so $p_0 = 1_{\sigma(S)}$ and $p_1 = \mathrm{id}_{\sigma(S)}$.

12.5. Theorem: Let $S \in \mathcal{L}(H)$ be self-adjoint. There exists a unique continuous linear map $\Phi : C(\sigma(S)) \to \mathcal{L}(H)$ such that

$$\Phi(p_0) = I, \quad \Phi(p_1) = S, \quad \Phi(f \cdot g) = \Phi(f)\Phi(g), \quad f, g \in C(\sigma(S)),$$

[Φ is multiplicative, i.e. an *algebra homomorphism*.]

For any $f \in C(\sigma(S))$ the operator $\Phi(f)$ is normal,

$$\Phi(f)^* = \Phi(\overline{f}),$$

and $\Phi(f)$ is self-adjoint if and only if f is real-valued.

Moreover, Φ is an isometry, i.e. $\|\Phi(f)\|_{\mathcal{L}(H)} = \|f\|_{\infty,\sigma(S)}$ for all $f \in C(\sigma(S))$.

Proof. For polynomials $p(\lambda) = \sum_{j=0}^{n} a_j \lambda^j$ we define $\Phi(p) := \sum_{j=0}^{n} a_j S^j$, and let $\mathcal{P}$ the algebra of all polynomial functions $p : \sigma(S) \to \mathbb{C}$. Then $\Phi : \mathcal{P} \to \mathcal{L}(H)$ is an algebra homomorphism, $\Phi(p)^* = \Phi(\overline{p})$ and $\Phi(p)$ is normal for each $p \in \mathcal{P}$. Moreover, $\Phi(p)$ is self-adjoint if p is real-valued. Clearly, $\Phi(p_0) = I$ and $\Phi(p_1) = S$. end Tue

By Weierstraß, $\mathcal{P}$ is dense in $C(\sigma(S))$ for the sup-norm $\|\cdot\|_\infty$. So we only have to check that $\Phi : \mathcal{P} \to \mathcal{L}(H)$ is an isometry. For $p \in \mathcal{P}$ we have 11.06.19

$$\begin{aligned} \|\Phi(p)\|^2 &= \|\Phi(p)^*\Phi(p)\| = \|\Phi(\overline{p})\Phi(p)\| = \|\Phi(\overline{p}p)\| = r(\Phi(\overline{p}p)) \\ &= \max\{|\lambda| : \lambda \in \sigma(\Phi(\overline{p}p))\} = \max\{|\overline{p}p(\mu)| : \mu \in \sigma(S)\} = \|p\|_{\infty,\sigma(S)}^2, \end{aligned}$$

where we used the polynomial spectral mapping result from the Exercise 23. If $\Phi(p)$ is self-adjoint, then we obtain $\Phi(\overline{p}) = \Phi(p)^* = \Phi(p)$, from which $\overline{p} = p$, i.e. real-valuedness of p, follows by injectivity of Φ.

The properties of Φ are preserved when approximating continuous functions by polynomials. $\square$

As an application of 12.5 we show "optimality" of self-adjoint operators with respect to resolvent bounds.

12.6. Corollary: Let $S \in \mathcal{L}(H)$ be self-adjoint. Then, for any $\lambda \in \rho(S)$,

$$R(\lambda, S) = \Phi((\lambda - (\cdot))^{-1}) \quad \text{and} \quad \|R(\lambda, S)\| = \frac{1}{d(\lambda, \sigma(S))}.$$

Proof. We have

$$(\lambda - S)\Phi((\lambda - (\cdot))^{-1}) = \Phi(\lambda - (\cdot))\Phi((\lambda - (\cdot))^{-1}) = \Phi(1_{\sigma(S)}) = I,$$

and $\Phi((\lambda - (\cdot))^{-1})(\lambda - S) = I$ is proved similarly. Recall that Φ is isometric and observe $\|(\lambda - (\cdot))^{-1}\|_{\infty,\sigma(S)} = d(\lambda, \sigma(S))^{-1}$. $\square$

12.7. Remark: Let $S \in \mathcal{L}(H)$ be self-adjoint. Then the functional calculus Φ from 12.5 is an extension of the Dunford functional calculus: if F is holomorphic on a complex neighborhood of $\sigma(S)$ then

$$\Phi(F|_{\sigma(S)}) = \frac{1}{2\pi i}\int_\Gamma F(\lambda)R(\lambda, S)\, d\lambda,$$

where Γ is suitably chosen ($\rightarrow$ 9.3): By 12.6 we have $R(\lambda, S) = \Phi((\lambda - (\cdot))^{-1})$, and $\Gamma \to \sigma(S)$, $\lambda \mapsto (\lambda - (\cdot))^{-1}$, is continuous. Hence we can interchange Φ with the integral:

$$\frac{1}{2\pi i}\int_\Gamma F(\lambda)R(\lambda, S)\, d\lambda = \frac{1}{2\pi i}\int_\Gamma F(\lambda)\Phi((\lambda-(\cdot))^{-1})\, d\lambda = \Phi\Big(\frac{1}{2\pi i}\int_\Gamma \frac{F(\lambda)}{\lambda - (\cdot)}\, d\lambda\Big) = \Phi(F|_{\sigma(S)}).$$

12.8. Definition: We call a self-adjoint operator $S \in \mathcal{L}(H)$ *positive* and write $S \geq 0$ if $(Sx|x) \geq 0$ for all $x \in H$. If $T \in \mathcal{L}(H)$ is another self-adjoint operator then $S \geq T$ means $S - T \geq 0$, i.e. $(Sx|x) \geq (Tx|x)$ for all $x \in H$.

12.9. Lemma: Let $S \in \mathcal{L}(H)$ be self-adjoint and $\Phi : C(\sigma(S)) \to \mathcal{L}(H)$ be the functional calculus for S from 12.5. If $f, g \in C(\sigma(S))$ are real-valued and $f \geq g$ then $\Phi(f) \geq \Phi(g)$ in the sense of 12.8.

Proof. It suffices to prove the case $g = 0$, i.e. $f \geq 0$. Then $\sqrt{f}$ is well defined, $\sqrt{f} \in C(\sigma(S))$, and $\Phi(f) = \Phi(\sqrt{f}\sqrt{f}) = \Phi(\sqrt{f})\Phi(\sqrt{f})$, where $\Phi(\sqrt{f})$ is selfadjoint. This implies, for $x \in H$,

$$(\Phi(f)x|x) = (\Phi(\sqrt{f})\Phi(\sqrt{f})x|x) = (\Phi(\sqrt{f})x|\Phi(\sqrt{f})x) \geq 0.$$

□

Before turning to the study of self-adjoint compact operators we give a lemma on general self-adjoint operators. Recall

- $x \perp y :\Longleftrightarrow (x|y) = 0$ for $x, y \in H$,
- $M \perp N :\Longleftrightarrow \forall x \in M, y \in N$: $(x|y) = 0$ for linear subspaces $M, N \subset H$,
- $M^\perp := \{x \in H : x \perp M\} = \{x \in H : \forall y \in M : x \perp y\}$ for linear subspaces M of H.

If M is a linear subspace of H then $M^\perp$ is always closed, $\overline{M}^\perp = M^\perp$, and $(M^\perp)^\perp = \overline{M}$. One has always $H = \overline{M} \oplus M^\perp$ where $\overline{M} \perp M^\perp$.

12.10. Lemma: Let $S \in \mathcal{L}(H)$ be self-adjoint. Then:

(a) For $\lambda, \mu \in \sigma(S)$ with $\lambda \neq \mu$: $N(\lambda - S) \perp N(\mu - S)$.

(b) For $\lambda \in \sigma(S)$: $N((\lambda - S)^2) = N(\lambda - S)$.

(c) The space H is the orthogonal direct sum of $N(S)$ and $\overline{R(S)}$, i.e. $H = N(S) \oplus \overline{R(S)}$ and $N(S) \perp \overline{R(S)}$.

Proof. (a) For $x \in N(\lambda - S)$ and $y \in N(\mu - S)$ we have

$$\lambda(x|y) = (\lambda x|y) = (Sx|y) = (x|Sy) = \mu(x|y),$$

and $x \perp y$, since $\lambda \neq \mu$.

(b) Let $x \in N((\lambda - S)^2)$. Then

$$\|(\lambda - S)x\|^2 = ((\lambda - S)x|(\lambda - S)x) = (x|\underbrace{(\lambda - S)^2 x}_{=0}) = 0,$$

i.e. $(\lambda - S)x = 0$, and $x \in N(\lambda - S)$. The reverse inclusion is clear.

(c) Let $x \in N(S)$ and $Sy \in R(S)$. Then $(x|Sy) = (Sx|y) = 0$, hence $N(S) \perp \overline{R(S)}$. If, on the other hand, $x \in H$ such that $x \perp \overline{R(S)}$ then

$$\|Sx\|^2 = (Sx|Sx) = (x|\underbrace{S^2 x}_{\in R(S)}) = 0,$$

i.e. $Sx = 0$ and $x \in N(S)$. Hence $N(S) = \overline{R(S)}^\perp$. □

12.11. Theorem: Let $S \in \mathcal{L}(H)$ be compact and self-adjoint and $\dim H = \infty$. There exists a real sequence $(\lambda_n)_{n\in\mathbb{N}_0}$ with $\lambda_n \to 0$ and an orthonormal sequence $(e_n)_{n\in\mathbb{N}_0}$ in H such that

$$S = \sum_{n=0}^{\infty} \lambda(\cdot|e_n)e_n,$$

where the series converges in operator norm.

Proof. We apply Theorem 11.9 to S. By 12.10(b) we know that $p = 1$ for every $\lambda \in \sigma(S) \setminus \{0\}$. We also know that $\sigma(S)$ consists of a null sequence. For any $\lambda \in \sigma(S) \setminus \{0\}$ we choose a finite orthonormal basis of $N(\lambda - S)$. We obtain thus an orthonormal sequence $(e_n)_{n\in N}$ where we assume that the corresponding eigenvalues are ordered such that $|\lambda_1| \geq |\lambda_2| \geq \ldots$. Here $N = \{0, \ldots, n_0\}$ is finite (if $\sigma(S) \setminus \{0\}$ is finite) or $N = \mathbb{N}_0$.

We let $P := \sum_{n\in N}(\cdot|e_n)e_n$. Then P is the orthogonal projection onto $H_1 := \overline{\text{span}\{e_n : n \in N\}}$. Letting $H_0 := H_1^{\perp}$ we have, for $x \in H_0$ and $n \in N$,

$$(Sx|e_n) = (x|Se_n) = \lambda_n(x|e_n) = 0.$$

i.e. $Sx \in H_0$. This means that $S_0 := S|_{H_0} \in \mathcal{L}(H_0)$. Clearly, S_0 is self-adjoint and compact. By 11.9, $\sigma(S_0) \setminus \{0\} = \emptyset$, hence $S_0 = 0$ by 12.2(b), and $H_0 \subset N(S)$. On the other hand, $N(S)$ is by 12.10(a) orthogonal to each $N(\lambda_n - S)$, $n \in N$. Hence $N(S) \subset H_0$, and we obtain $H_0 = N(S)$, and by 12.10(c) also $H_1 = \overline{R(S)}$.

But then

$$Sx = SPx = \sum_{n\in N} \lambda_n(x|e_n)e_n \quad \text{for all } x \in H.$$

If $N = \{0, \ldots, n_0\}$ is finite, we set $\lambda_n = 0$ for $n > n_0$, and choose an orthonormal sequence $(e_n)_{n>n_0}$ in $N(S)$.

For $x \in H$ and $k \in \mathbb{N}$ we have, by Pythagoras and Bessel's inequality,

$$\|Sx - \sum_{n=0}^{k} \lambda_n(x|e_n)e_n\|^2 = \sum_{n>k} |\lambda_n(x|e_n)|^2 \leq \|x\|^2(\sup_{n>k} |\lambda_n|)^2,$$

which proves convergence of the series in operator norm, since (λ_n) is a null sequence. □

end Thu 13.06.19

12.12. Theorem: Let G be another Hilbert space with $\dim G = \infty$ and $T \in \mathcal{K}(H, G)$. Then there exists a decreasing null sequence $(s_n)_{n\in\mathbb{N}_0}$ in $[0, \infty)$ and orthonormal systems $(e_n)_{n\in\mathbb{N}_0}$ in H and $(f_n)_{n\in\mathbb{N}_0}$ in G such that

$$T = \sum_{n=0}^{\infty} s_n(\cdot|e_n)f_n,$$

where the series converges in operator norm.

Proof. The operator $T^*T \in \mathcal{L}(H)$ is compact and self-adjoint. Moreover, $T^*T \geq 0$ in the sense of 12.8, and $\sigma(T^*T) \subseteq [0, \|T\|^2]$ by 12.4. By 12.11 we obtain a decreasing null sequence $(s_n)_{n\in\mathbb{N}_0}$ and an orthonormal system $(e_n)_{n\in\mathbb{N}_0}$ in H such that

$$T^*T = \sum_{n=0}^{\infty} s_n^2(\cdot|e_n)e_n.$$

For $n \in \mathbb{N}_0$ with $s_n > 0$ we let $f_n := s_n^{-1}Te_n$. For $n, m \in \mathbb{N}_0$ with $s_n s_m > 0$ we then have

$$(f_n|f_m) = (s_n s_m)^{-1}(Te_n|Te_m) = (s_n s_m)^{-1}(T^*Te_n|e_m) = \frac{s_n^2}{s_n s_m}(e_n|e_m) = \delta_{nm}.$$

If $N = \{n \in \mathbb{N}_0 : s_n > 0\}$ is finite then we extend $(f_n)_{n\in N}$ to an orthonormal sequence $(f_n)_{n\in\mathbb{N}_0}$ in G. If $y \perp e_n$ for all $n \in N$ then

$$\|Ty\|^2 = (T^*Ty|y) = 0$$

by the representation of T^*T. Hence, for any $x \in H$,

$$\begin{aligned} Tx &= T\Big(\underbrace{x - \sum_{n\in N}(x|e_n)e_n}_{\in N(T)}\Big) + T\Big(\sum_{n\in N}(x|e_n)e_n\Big) \\ &= \sum_{n\in N}(x|e_n)Te_n = \sum_{n\in N} s_n(x|e_n)f_n = \sum_{n=0}^{\infty} s_n(x|e_n)f_n. \end{aligned}$$

The proof for convergence in operator norm is the same as in 12.11. □

12.13. Corollary: Let G be another Hilbert space and

$$\mathcal{F}(H,G) := \{T \in \mathcal{L}(H,G) : \dim R(T) < \infty\}$$

denote the space of *finite rank operators* from H to G. Then $\mathcal{F}(H,G)$ is dense in $\mathcal{K}(H,G)$ with respect to operator norm.

Remark: This result is in general false in the Banach space context (by an example of Enflo 1973). The *rank* of an operator in the dimension of its range.

12.14. Definition and Remark: A representation of $T \in \mathcal{K}(H,G)$ as a series with the properties of 12.12 is called a *Schmidt representation* of T. The sequence $(s_n)_{n\in\mathbb{N}_0}$ is uniquely determined by T (as the decreasing eigenvalue sequence of T^*T), but in general not the orthonormal systems (e_n) and (f_n). The sequence $(s_n)_{n\in\mathbb{N}_0}$ is called the sequence of *singular values* of the operator T and denoted by $(s_n(T))_{n\in\mathbb{N}_0}$. The following gives a characterization of the singular values of an operator as *approximation numbers*:

For $T \in \mathcal{K}(H,G)$ and $n \in \mathbb{N}_0$ one has

$$s_n(T) = \inf\{\|T - U\| : U \in \mathcal{L}(H,G), \dim R(U) \leq n\} =: \alpha_n(T).$$

Here $\alpha_n(T)$ measures how good T can be approximated by operators of rank of at most n.

Proof. Let $\sum_{j=0}^{\infty} s_j(\cdot|e_j)f_j$ be a Schmidt representation of T. Then, for $n \in \mathbb{N}_0$ and $x \in H$, we have by Bessel's inequality

$$\|Tx - \sum_{j=0}^{n-1} s_j(x|e_j)f_j\|^2 \le \sum_{j=n}^{\infty} s_j^2 |(x|e_j)|^2 \le s_n^2 \|x\|^2.$$

This implies that $\alpha_n(T) \le s_n$.

Now let $U \in \mathcal{L}(H, G)$ with $\dim R(U) \le n$. The restriction of U to the $(n+1)$-dimensional space $\operatorname{lin}\{e_0, \ldots, e_n\}$ has non-trivial kernel, so we find $y = \sum_{j=0}^{n} \xi_j e_j$ with $\|y\| = 1$ and $Uy = 0$. By Pythagoras we thus obtain

$$\|T - U\|^2 \ge \|(T-U)y\|^2 = \|Ty\|^2 = \|\sum_{j=0}^{n} s_j \xi_j f_j\|^2 = \sum_{j=0}^{n} s_j^2 |\xi_j|^2 \ge s_n^2 \sum_{j=0}^{n} |\xi_j|^2 = s_n^2.$$

Hence $\alpha_n(T) \ge s_n$. □

Remark: For $1 \le p < \infty$ the *Schatten p-class* is defined by

$$S_p(H, G) := \{T \in \mathcal{K}(H, G) : (s_n(T))_{n \in \mathbb{N}_0} \in l^p\}$$

and $\nu_p(T) := (\sum_{n=0}^{\infty} s_n(T)^p)^{1/p}$ for $T \in S_p(H, G)$. Elements of $S_2(H, G)$ are called *Hilbert-Schmidt opertors* and elements of $S_1(H, G)$ are called *nuclear operators* or, for $G = H$, are said to be of *trace class*.

In many respects, the spaces $S_p(H, G)$ may be viewed as "non-commutative" analogs of the spaces l^p (cf., e.g., §16 in Meise/Vogt "Introduction to Functional Analysis").

Remark: If H and G are separable infinite-dimensional Hilbert spaces, then the sequences $(e_n)_{n \in \mathbb{N}_0}$ and $(f_n)_{n \in \mathbb{N}_0}$ in 12.11 and 12.12 can be chosen to be orthonormal bases of H and G, respectively. This can be seen from the proof.

A related observation is that, for a compact operator $T \in \mathcal{K}(H, G)$, the space $\overline{R(T)}$ is always separable. Moreover, if $T \in \mathcal{K}(H)$ is self-adjoint and injective, then the space H is separable (since the space H_0 in the proof is trivial).

13 The spectral theorem for self-adjoint operators

For a self-adjoint operator $S \in \mathcal{L}(H)$, we want to extend the functional calculus of Theorem 12.5 to a functional calculus for bounded Borel measurable functions on $\sigma(S)$.

13.1. Definition: For any compact subset $K \subseteq \mathbb{R}$ we let

$$\mathcal{B}_b(K) := \{f : K \to \mathbb{C} : \ f \text{ is Borel measurable and bounded}\},$$

equipped with the norm $\|f\|_\infty := \sup_{t \in K} |f(t)|$ (notice, that $\mathcal{B}_b(K)$ is a space of functions, not a space of co-classes of functions!). Then $\mathcal{B}_b(K)$ is a Banach space.

If (f_n) is a sequence of functions $M \to \mathbb{C}$ and $f : M \to \mathbb{C}$ is a function, then we write $f_n \to f$ *bounded pointwise*, if $f_n \to f$ pointwise and $\sup_n \|f_n\|_\infty < \infty$. Observe that $\mathcal{B}_b(K)$ is closed under bounded pointwise convergence.

We call a map $\Psi : \mathcal{B}_b(K) \to \mathcal{L}(H)$ *σ-continuous*, if $f_n \to f$ bounded pointwise implies $(\Psi(f_n)x|y) \to (\Psi(f)x|y)$ for all $x, y \in H$.

13.2. Lemma: Let $K \subseteq \mathbb{R}$ be compact. Then $\mathcal{B}_b(K)$ is the smallest subset M of $\mathbb{C}^K$ satisfying

(1) $C(K) \subseteq M$,

(2) M is closed under bounded pointwise convergence.

end Tue 18.06.19

Proof. Let M_0 be the smallest subset M of $\mathbb{C}^K$ such that (1) and (2) hold, i.e. the intersection of all subsets M of $\mathbb{C}^K$ with (1) and (2). Since $\mathcal{B}_b(K)$ satisfies (1) and (2), M_0 is well-defined and we have $M_0 \subseteq \mathcal{B}_b(K)$.

For $\lambda \in \mathbb{C} \setminus \{0\}$, $\frac{1}{\lambda} M_0$ satisfies (1) and (2), hence $M_0 \subseteq \frac{1}{\lambda} M_0$, i.e. $\lambda M_0 \subseteq M_0$.

If $f \in C(K)$, then $M_0 - f$ satisfies (1) and (2), thus $M_0 \subseteq M_0 - f$, i.e. $f + M_0 \subseteq M_0$. Hence $C(K) + M_0 \subseteq M_0$. Now let $S := \{f : f + M_0 \subseteq M_0\}$. We just have shown $C(K) \subseteq S$. If (f_n) is a sequence in S with $f_n \to f$ bounded pointwise and if $g \in M_0$, then $(f_n + g)$ is a sequence in M_0 (by $f_n \in S$) and $f_n + g \to f + g$ bounded pointwise. By (2) for M_0 we have $f + g \in M_0$. Since $g \in M_0$ was arbitrary, we obtain $f \in S$, and have shown (2) for S. This yields $M_0 \subseteq S$, i.e. $M_0 + M_0 \subseteq M_0$.

We thus have shown that M_0 is a complex vector space. Moreover, $M := \{g : |g| \in M_0\}$ satisfies (1) and (2), hence $M_0 \subseteq M$, i.e. $f \in M_0$ implies $|f| \in M_0$. Since M_0 is a vector space we obtain

$$\max\{f, g\}, \min\{f, g\} = \frac{1}{2}((f + g) \pm |f - g|) \in M_0$$

for real-valued $f, g \in M_0$.

Now we let $\mathcal{F} := \{A \subseteq K : 1_A \in M_0\}$. Then $\mathcal{F}$ is a σ-algebra in K: $\emptyset, K \in \mathcal{F}$ is clear by $0, 1_K \in C(K) \subseteq M_0$. If $A \in \mathcal{F}$, then $K \setminus A \in \mathcal{F}$, since M_0 is a vector space. If $(A_j)_{j\in\mathbb{N}}$ is a sequence in $\mathcal{F}$, then $\bigcup_{j=1}^n A_j \in \mathcal{F}$, since the max of a finite family of functions in M_0 belongs to M_0. Finally $\bigcup_{j\in\mathbb{N}} A_j \in \mathcal{F}$ follows from (2) for M_0.

Also by (2), $\mathcal{F}$ contains all sets $(-\infty, b] \cap K$, $b \in \mathbb{R}$, hence all Borel subsets of K. Since each $f \in \mathcal{B}_b(K)$ can be uniformly on K approximated by a sequence (f_n) of simple Borel functions, we obtain $\mathcal{B}_b(K) \subseteq M_0$. □

13.3. Lemma: Let $K \subseteq \mathbb{R}$ be compact. If $\Phi, \Psi : \mathcal{B}_b(K) \to \mathcal{L}(H)$ are σ-continuous and coincide on $C(K)$ then $\Phi = \Psi$.

Proof. Let

$$M := \{f \in \mathcal{B}_b(K) : \Phi(f) = \Psi(f)\} = \{f \in \mathcal{B}_b(K) : \forall x, y \in H : (\Phi(f)x|y) = (\Psi(f)x|y)\}.$$

Then M satisfies (1), (2) from 13.2, hence $M = \mathcal{B}_b(K)$, i.e. $\Phi = \Psi$. □

The following proposition is essential for our proof of the Spectral Theorem. We postpone the proof of this proposition for now.

13.4. Proposition: Let $K \subseteq \mathbb{R}$ be compact and $\phi \in C(K)'$. Then there exists a unique σ-continuous extension $\widetilde{\phi} \in \mathcal{B}_b(K)'$ of ϕ. Moreover, we have $\|\widetilde{\phi}\| = \|\phi\|$.

We shall also use the following lemma on the relation of continuous sesquilinear forms and linear operators.

13.5. Lemma: Let $\beta : H \times H \to \mathbb{C}$ be *sesquilinear* (i.e. linear in the first and antilinear in the second component) and $M \geq 0$ such that

$$|\beta(x, y)| \leq M\|x\|\|y\| \quad \text{for all } x, y \in H.$$

Then there exists a unique linear operator $B \in \mathcal{L}(H)$ such that

$$\beta(x, y) = (Bx|y) \quad \text{for all } x, y \in H,$$

and one has $\|B\| \leq M$.

Proof. For any $x \in H$, we have $\overline{\beta(x, \cdot)} \in H'$ and define $Bx := J_H^{-1}(\overline{\beta(x, \cdot)})$. Then B is linear (since it is a composition of two antilinear maps), and $\|Bx\| \leq M\|x\|$ for all $x \in H$, i.e. $B \in \mathcal{L}(H)$ and $\|B\| \leq M$. For $x, y \in H$ we have

$$(Bx|y) = \overline{(y|Bx)} = \overline{(J_H(Bx))(y)} = \overline{\overline{\beta(x, \cdot)}(y)} = \beta(x, y).$$

Uniqueness of B is clear. □

We now present the Spectral Theorem for bounded self-adjoint operators.

13.6. Theorem (Functional calculus for bounded Borel measurable functions): Let $S \in \mathcal{L}(H)$ be self adjoint. Then there exists a unique σ-continuous extension $\Psi : \mathcal{B}_b(\sigma(S)) \to \mathcal{L}(H)$ of the functional calculus $\Phi : C(\sigma(S)) \to \mathcal{L}(H)$ from 12.5. The map Ψ is linear and multiplicative, i.e. an algebra homomorphism, and satisfies $\|\Psi\| = \|\Phi\| = 1$. Moreover, we have

(i) for each $f \in \mathcal{B}_b(\sigma(S))$ the operator $\Psi(f)$ is normal and $\Psi(\overline{f}) = \Psi(f)^*$,

(ii) for real-valued $f \in \mathcal{B}_b(\sigma(S))$ the operator $\Psi(f)$ is self-adjoint,

(iii) for $f \geq 0$ we have $\Psi(f) \geq 0$ in the sense of 12.8,

(iv) if $f_n \to f$ bounded pointwise then $\Psi(f_n)x \to \Psi(f)x$ for all $x \in H$.

Warning: We have $\|\Phi(f)\| = \|f\|_\infty$ for $f \in C(\sigma(S))$ and $\|\Psi(g)\| \leq \|g\|_\infty$ for all $g \in \mathcal{B}_b(\sigma(S))$, but it may happen that $\|\Psi(g)\| < \|g\|_\infty$ for some $g \in \mathcal{B}_b(\sigma(S))$: for $\lambda \in \sigma(S)$ one has $\Psi(1_{\{\lambda\}}) = 0$ if and only if $\lambda \notin \sigma_p(S)$. ($\to$ exercises).

Proof. Uniqueness follows from 13.3. For the existence proof, we put, for $x, y \in H$,

$$\phi_{x,y} : C(\sigma(K)) \to \mathbb{C},\ f \mapsto \phi_{x,y}(f) := (\Phi(f)x|y).$$

Then $\phi_{x,y} \in C(\sigma(S))'$ and $\|\phi_{x,y}\| \leq \|x\|\|y\|$. Moreover, the map $H \times H \mapsto C(\sigma(S))'$ is sesquilinear.

By 13.4 we find for each pair $(x, y) \in H \times H$ a unique σ-continuous extension $\widetilde{\phi}_{x,y} : \mathcal{B}_b(\sigma(S)) \to \mathbb{C}$ and $\|\widetilde{\phi}_{x,y}\| = \|\phi_{x,y}\| \leq \|x\|\|y\|$. We show that $(x, y) \mapsto \widetilde{\phi}_{x,y}$ is sesquilinear: For $\alpha \in \mathbb{C}$ and $x, y, z \in H$ the σ-continuous funkcionals $\widetilde{\phi}_{\alpha x+y,z}$ and $\alpha\widetilde{\phi}_{x,z} + \widetilde{\phi}_{y,z}$ or $\widetilde{\phi}_{x,\alpha y+z}$ und $\overline{\alpha}\widetilde{\phi}_{x,y} + \widetilde{\mu}_{x,z}$, respectively, coincide on $C(\sigma(S))$. By 13.3 they coincide on $\mathcal{B}_b(\sigma(S))$.

For each $f \in \mathcal{B}_b(\sigma(S))$ we define

$$\beta_f : H \times H \to \mathbb{C},\ (x, y) \mapsto \beta_f(x, y) := \widetilde{\phi}_{x,y}(f).$$

Then

$$|\beta_f(x, y)| = |\widetilde{\phi}_{x,y}(f)| \leq \|f\|_\infty \|x\|\|y\| \quad \text{for all } x, y \in H,\ f \in \mathcal{B}_b(\sigma(S)).$$

Moreover, each β_f is sesquilinear. By 13.5 we find, for each $f \in \mathcal{B}_b(\sigma(S))$, a unique operator operator $\Psi(f) \in \mathcal{L}(H)$ such that

$$\widetilde{\phi}_{x,y}(f) = \beta_f(x, y) = (\Psi(f)x|y) \quad \text{for all } x, y \in H.$$

We have $\|\Psi(f)\| \leq \|f\|_\infty$. Clearly $\Psi(f) = \Phi(f)$ for all $f \in C(\sigma(S))$. By construction, $\Psi : f \mapsto \Psi(f)$ auf $\mathcal{B}_b(\sigma(S))$ is σ-continuous.

We show that $f \mapsto \Psi(f)$ is linear: For $\alpha \in \mathbb{C}$ and $f, g \in \mathcal{B}_b(\sigma(S))$ we have

$$\begin{aligned}((\alpha\Psi(f) + \Psi(g))x|y) &= \alpha(\Psi(f)x|y) + (\Psi(g)x|y) = \alpha\widetilde{\phi}_{x,y}(f) + \widetilde{\phi}_{x,y}(g) = \widetilde{\phi}_{x,y}(\alpha f + g) \\ &= (\Psi(\alpha f + g)x|y),\end{aligned}$$

for all $x, y \in H$. By 13.5 (uniqueness) we thus obtain $\alpha\Psi(f) + \Psi(g) = \Psi(\alpha f + g)$, i.e. $\Psi : f \mapsto \Psi(f)$ is linear.

For $f \in C(\sigma(S))$ the map $g \mapsto \Psi(fg) - \Psi(f)\Psi(g)$ is σ-continuous and vanishes on $C(\sigma(S))$, by 13.3 it thus vanishes on $\mathcal{B}_b(\sigma(S))$. We obtain $\Psi(fg) = \Psi(f)\Psi(g)$ for all $g \in \mathcal{B}_b(\sigma(S))$. For fixed $g \in \mathcal{B}_b(\sigma(S))$ the σ-continuous map $f \mapsto \Psi(fg) - \Psi(f)\Psi(g)$ vanishes on $C(\sigma(S))$, hence on $\mathcal{B}_b(\sigma(S))$ by 13.3. We obtain $\Psi(fg) = \Psi(f)\Psi(g)$ for all $f, g \in \mathcal{B}_b(\sigma(S))$, i.e. Ψ is multiplicative.

Now we prove (i). Since $f \mapsto \Psi(\overline{f}) - \Psi(f)^*$ is σ-continuous on $\mathcal{B}_b(\sigma(S))$ and vanishes on $C(\sigma(S))$, it vanishes by 13.3 on $\mathcal{B}_b(K)$. Hence we have $\Psi(\overline{f}) = \Psi(f)^*$ for all $f \in \mathcal{B}_b(\sigma(S))$. For each $f \in \mathcal{B}_b(\sigma(S))$ we then have

$$\Psi(f)\Psi(f)^* = \Psi(f)\Psi(\overline{f}) = \Psi(|f|^2) = \Psi(\overline{f})\Psi(f) = \Psi(f)^*\Psi(f),$$

so $\Psi(f)$ is normal.

We prove (ii). For real-valued f we have

$$\Psi(f)^* = \Psi(\overline{f}) = \Psi(f),$$

i.e. $\Psi(f)$ is self-adjoint. For the proof of (iii) let $f \geq 0$, then $g := \sqrt{f} \in \mathcal{B}_b(\sigma(S))$ and for $x \in H$ we have

$$(\Psi(f)x|x) = (\Psi(g^2)x|x) = (\Psi(g)\Psi(g)x|x) = (\Psi(g)x|\Psi(g)x) \geq 0,$$

where we used that $\Psi(g)$ is self-adjoint. For the proof of (iv) it suffices to study the case $f = 0$. So let $f_n \to 0$ bounded pointwise and $x \in H$. We then have

$$\|\Psi(f_n)x\|^2 = (\Psi(f_n)^*\Psi(f_n)x|x) = (\Psi(|f_n|^2)x|x) \longrightarrow 0,$$

by σ-continuity since $|f_n|^2 \to 0$ bounded pointwise. □

We still have to prove Proposition 13.4.

Proof of Proposition 13.4. Uniqueness follows from 13.3. We assume $K \neq \emptyset$ and prove existence. We first restrict to the case of intervals. Letting $a := \min K$ and $b := \max K$ we have $K \subseteq [a, b]$ and $[a, b] \setminus K$ is open (in $[a, b]$ but also in $\mathbb{R}$!). Hence $[a, b] \setminus K$ is an at most countable disjoint union of open intervals $I_j = (a_j, b_j)$ with $a_j, b_j \in K$. Let R denote the restriction map $R : C[a, b] \to C(K)$, $f \mapsto f|_K$ and define the extension operator $J : \mathcal{B}_b(K) \to \mathcal{B}_b[a, b]$ by $Jf(x) = f(x)$ for $x \in K$ and $Jf(x) := \frac{x-a_j}{b_j-a_j} f(a_j) + \frac{b_j-x}{b_j-a_j} f(b_j)$ for $x \in I_j$. Then $\|Jf\|_\infty = \|f\|_\infty$ for all f, $J(C(K)) \subseteq C[a, b]$ and $RJ = I_{C(K)}$. If $\phi \in C(K)'$

and ϕR has a σ-continuous extension $\widetilde{\psi} \in \mathcal{B}_b[a,b]'$ then $\widetilde{\phi} := \widetilde{\psi} J \in \mathcal{B}_b(K)'$ is a σ-continuous extension of ϕ.

We may thus assume that $K = [a,b]$ and $\phi \in C[a,b]'$. Now we use FA 3.10 and find a function $g \in BV_0[a,b]$ with $\|g\|_{BV} = \|\phi\|$ such that $\phi(f) = \int_a^b f(t)\,dg(t)$ as a Stieltjes integral for any $f \in C[a,b]$, where we may assume that g is continuous from the right. Any real-valued BV-function is the difference of two monotone increasing functions, so we find monotone increasing and right continuous functions $g_j : [a,b] \to$, $j = 1,\ldots,4$, with $g_j(0) = 0$ and

$$g(t) = g_1(t) - g_2(t) + i(g_3(t) - g_4(t)), \quad t \in [a,b].$$

We extend each g_j to a function on $\mathbb{R}$ by letting $g_j(t) = 0$ for $t < a$ and $g_j(t) = g_j(b)$ for $t > b$. Then $\mu_j((c,d]) := g_j(d) - g_j(c)$ induces a positive and finite Borel measure μ_j on $\mathbb{R}$, for which we have

$$\int f\,d\mu_1 - \int f\,d\mu_2 + i\Big(\int f\,d\mu_3 - \int f\,d\mu_4\Big) = \int_a^b f(t)\,dg(t), \quad f \in C[a,b].$$

For each j, the map $f \mapsto \int f\,d\mu_j \in \mathcal{B}_b[a,b]'$ is a σ-continuous extension of $f \mapsto \int_a^b g(t)\,dg_j(t)$. Hence

$$f \mapsto \int f\,d\mu_1 - \int f\,d\mu_2 + i\Big(\int f\,d\mu_3 - \int f\,d\mu_4\Big) \in \mathcal{B}_b[a,b]'$$

is a σ-continuous extension of ϕ. One also has $|\int f\,d\mu| \le \|f\|_\infty |\mu|([a,b])$ and $|\mu|([a,b]) = \|g\|_{BV}$, but we do not go into details here.[4] $\square$

end Thu 27.06.19

13.7. Definition: Let A be a linear operator in H, i.e. $A : H \supseteq D(A) \to H$ is linear.

(i) A is called *symmetric* if $(Ax|y) = (x|Ay)$ for all $x, y \in D(A)$.

(ii) If A is densely defined then A is called self-adjoint if $A = A^*$.

Recall that, in the situation of (ii), the adjoint operator A^* of A is given by

$$x \in D(A^*) \text{ and } A^*x = y \iff \forall z \in D(A) : (Az|x) = (z|y).$$

Remark: If A is densely defined in H then A is symmetric if and only if $A \subseteq A^*$. Here, $A \subseteq A^*$ means that $x \in D(A)$ implies $x \in D(A^*)$ and $A^*x = Ax$.

In particular, any self-adjoint operator is symmetric. Moreover, any self-adjoint operator is closed.

Remark: If A is symmetric then $(Ax|x) \in \mathbb{R}$ for all $x \in D(A)$, since

$$(Ax|x) = (x|Ax) = \overline{(Ax|x)}.$$

[4] We refer to Rudin: Real and Complex Analysis.

13.8. Lemma: If A is densely defined and symmetric in H, then A is closable and its closure $\overline{A}$ is symmetric.

Proof. By the remark above we have $A \subseteq A^*$, and A^* is always closed. Hence also $\overline{A} \subseteq A^* = (\overline{A})^*$ (for the last identity see below). $\square$

Recall: In the situation of 13.8, we have by 8.17 and reflexivity of H:

$$A \text{ is closable} \iff A^* \text{ is densely defined.}$$

Actually, the proof shows that, in this case, $\overline{A} = (A^*)^*$. Applied to the closed operator A^* in place of A, we get

$$(\overline{A})^* = ((A^*)^*)^* = A^*.$$

13.9. Lemma: Let A be symmetric and closed. Then, for any $z \in \mathbb{C} \setminus \mathbb{R}$, the operator $z - A$ is injective and $R(z - A)$ is closed. If $R(z - A) = H$ then $\|R(z, A)\| \leq 1/|\mathrm{Im}\, z|$.

Proof. Since A is symmetric, we have for $x \in D(A)$ and $\xi + i\eta \in \mathbb{C}$ with $\eta \neq 0$ by the preceding remark

$$\begin{aligned} \|(\xi + i\eta - A)x\|^2 &= (\xi^2 + \eta^2)\|x\|^2 - 2\mathrm{Re}\,((\xi + i\eta)x|Ax) + \|Ax\|^2 \\ &= (\xi^2 + \eta^2)\|x\|^2 - 2\xi(x|Ax) + \|Ax\|^2 \\ &= \eta^2\|x\|^2 + \|(\xi - A)x\|^2 \geq \eta^2\|x\|^2. \end{aligned}$$

Since $\xi + i\eta - A$ is closed, the assertion follows. $\square$

13.10. Proposition: If A is a self-adjoint operator in H then $\sigma(A) \subset \mathbb{R}$, and $\|R(z, A)\| \leq |\mathrm{Im}\, z|^{-1}$ for any $z \in \mathbb{C} \setminus \mathbb{R}$.

Proof. We have $(z - A)^* = \overline{z} - A$ for any $z \in \mathbb{C}$. For $z \in \mathbb{C} \setminus \mathbb{R}$ we have by 13.9 that $z - A$, $\overline{z} - A$ are injective and that $R(z - A)$ is closed. But

$$R(z - A)^\perp = \{y \in H : \forall x \in D(A) : ((z - A)x|y) = 0\} = N((z - A)^*) = N(\overline{z} - A) = \{0\},$$

i.e. $R(z - A) = H$, and we obtain $z \in \rho(A)$. The resolvent estimate is from 13.9. $\square$

13.11. Proposition: For any densely defined operator in A the following are equivalent:

(i) A is closable and $\overline{A}$ is self-adjoint.

(ii) A is symmetric and $\sigma(\overline{A}) \subset \mathbb{R}$.

(iii) A is symmetric and there exists $z \in \mathbb{C} \setminus \mathbb{R}$ such that $z - A^*$ and $\overline{z} - A^*$ are injective.

Proof. (i) $\Longrightarrow$ (ii): If $\overline{A}$ is self-adjoint, then $\overline{A}$ and hence also A are symmetric. $\sigma(\overline{A}) \subset \mathbb{R}$ holds by 13.10.

(ii) $\Longrightarrow$ (iii): By 13.8, $\overline{A}$ is symmetric. Moreover, $(\overline{A})^* = A^*$. For $z \in \mathbb{C} \setminus \mathbb{R}$ we thus have $z, \overline{z} \in \sigma(A^*)$, and $z - A^*$, $\overline{z} - A^*$ are injective.

(iii) $\Longrightarrow$ (i): By 13.8, A is closable, and we have $\overline{A} \subset A^* = (\overline{A})^*$. By 13.9 (and its proof) the assumption implies $z, \overline{z} \in \rho(\overline{A})$, and this in turn implies $\overline{z}, z \in \rho(A^*)$. Hence we have $\rho(\overline{A}) \cap \rho(A^*) \neq \emptyset$ which by $\overline{A} \subset A^*$ is only possible if $A = A^*$. $\square$

Remark: Here we have used the following: If B, C are clsoed operators in a Banach space X such that $B \subset C$ and there exists $\lambda \in \rho(B) \cap \rho(C)$, then $B = C$.

For the proof observe that $\lambda - C : D(C) \to X$ is bijective so that $\lambda - B : D(B) \to X$ cannot be surjective if $D(B) \overset{\subset}{\neq} D(C)$.

13.12. Definition: A densely defined symmetric operator A in H is called *essentially self-adjoint* if its closure $\overline{A}$ is self-adjoint in H.

Remark: By 13.11, a densely defined and symmetric operator A is essentially self-adjoint if and only if the operators $i - A^*$ and $-i - A^*$ are both injective. We will get back to that later on.

We aim at an extension of the functional calculus from Theorem 13.6 to unbounded self-adjoint operators A. This will be done by an orthogonal decompopsition of H with respect to an auxiliary bounded self-adjoint operator B associated to A.

13.13. Lemma: Let A be self-ajoint in H and set

$$B := \frac{1}{2i}(R(i,A)^* - R(i,A)), \quad C := \frac{-1}{2}(R(i,A) + R(i,A)^*).$$

Then $B, C \in \mathcal{L}(H)$ are self-adjoint, $BA \subset AB = C$, B is injective and $0 \leq B \leq I$ (in the sense of 12.8).

Proof. Self-adjointness of B and C is clear. By $R(i,A)^* = R(-i,A)$ we obtain easily $BA \subseteq AB$. Moreover, since $AR(\lambda, A) = \lambda R(\lambda, A) - I$, we have

$$AB = \frac{1}{2i}(AR(-i,A) - AR(i,A)) = \frac{1}{2i}(-iR(-i,A) - iR(i,A)) = C.$$

By 13.9, $\|R(\pm i, A)\| \leq 1$, and we obtain $B \leq I$. For $x \in H$ and $y = R(i,A)x$ we have

$$(Bx|x) = \frac{1}{2i}((x|R(i,A)x) - (R(i,A)x|x)) = \operatorname{Im}(x|R(i,A)x) = \operatorname{Im}((i-A)y|y) = (y|y) \geq 0.$$

Hence $B \geq 0$. On the other hand, $Bx = 0$ implies $(i - A)x = y = 0$ and $x = 0$ by injectivity of $i - A$. Hence B is injective. $\square$

Remark: By the resolvent equation 8.6(a) we have

$$B = \frac{1}{2i}(R(-i,A) - R(i,A)) = \frac{1}{2i}((2i)R(i,A)R(-i,A)) = R(i,A)R(i,A)^*,$$

and the properties of B also follow from this representation.

We shall use the functional calculus Ψ from 13.6 for the operator B and write $f(B) := \Psi(f)$ for $f \in \mathcal{B}_b(\sigma(B))$. If f is defined on a superset of $\sigma(B)$ then we write $f(B) := (f|_{\sigma(B)})(B)$. Observe that $\sigma(B) \subseteq [0,1]$ by 13.13 and 12.3 and that $0 \notin \sigma_p(B)$ by 13.13.

13.14. Proposition: Let A be a self-adjoint operator in H, and let B, C be as in 13.13. For any $n \in \mathbb{N}$, define functions $\theta_n, s_n : \mathbb{R} \to \mathbb{R}$ by $\theta_n := 1_{(\frac{1}{n+1},\frac{1}{n}]}$ and $s_n(t) := \frac{1}{t}\theta_n(t)$, and let $P_n := \theta_n(B)$. Then:

(a) For each $n \in \mathbb{N}$, P_n is an orthogonal projection in H and

$$P_n A \subset AP_n = s_n(B)C \in \mathcal{L}(H).$$

(b) With $H_n := R(P_n)$ we have $H_n \subseteq D(A)$, $A(H_n) \subseteq H_n$ and $H_n \perp H_k$ for all $n,k \in \mathbb{N}$ with $n \neq k$. Moreover, $A_n := A|_{H_n} \in \mathcal{L}(H_n)$ is self-adjoint in the Hilbert space H_n.

(c) We have

$$x = \sum_{n\in\mathbb{N}} P_n x \quad \text{for all } x \in H \text{ (convergence in } H)$$

and

$$\begin{aligned} D(A) &= \{x \in H : \sum_{n\in\mathbb{N}} \|AP_n x\|^2 < \infty\} \\ Ax &= \sum_{n\in\mathbb{N}} AP_n x \quad \text{for all } x \in D(A) \text{ (convergence in } H). \end{aligned}$$

(d) For each $n \in \mathbb{N}$ we have

$$\sigma(A_n) \subseteq \sigma(A) \cap \left([-\sqrt{n}, -\sqrt{n-1}] \cup [\sqrt{n-1}, \sqrt{n}]\right).$$

Proof. (a) By $ts_n(t) = \theta_n(t)$, $t \in \mathbb{R}$, we get $Bs_n(B) = \theta_n(B) = P_n$. Hence, by 13.13,

$$AP_n = ABs_n(B) = Cs_n(B) \stackrel{(*)}{=} s_n(B)C \in \mathcal{L}(H)$$

(we say more on the equality $(*)$ below in the proof of (d)). Moreover, again by 13.13,

$$P_n A = Bs_n(B)A = s_n(B)BA \subseteq s_n(B)AB = s_n(B)C = AP_n.$$

By 13.6, we have $P_n^2 = P_n$ and $P_n^* = P_n$.

(b) Each H_n is a closed subspace of H and thus itself a Hilbert space. By (a), $AP_n \in \mathcal{L}(H)$, so $H_n \subset D(A)$ and, for $x \in H_n$:

$$P_nAx = AP_nx = Ax, \quad \text{i.e. } Ax \in H_n.$$

Hence $A(H_n) \subseteq H_n$. For $x \in H_n$, $y \in H_k$ and $n \neq k$ we have by 13.6:

$$(x|y) = (P_nx|P_ky) = (\underbrace{P_kP_n}_{=0}x|y) = 0,$$

and thus $H_n \perp H_k$. Finally, $A_n \in \mathcal{L}(H_n)$ is self-adjoint in H_n, since it inherits symmetry from A.

(c) We let $P_0 := 1_{\{0\}}(B)$ and $H_0 := P_0(H) = R(P_0)$. P_0 is an orthogonal projection and $BP_0 = (0 \cdot 1_{\{0\}})(B) = 0$ by 13.6, so $H_0 \subseteq N(B) = \{0\}$ by 13.13. Hence $P_0 = 0$.

Using 13.6, we thus obtain

$$I_H = 1_{[0,1]}(B) = 1_{(0,1]}(B) + P_0 = 1_{(0,1]}(B).$$

We have

$$\sum_{n=1}^{m} \theta_n = 1_{(\frac{1}{m+1},1]} \to 1_{(0,1]} \quad \text{bounded pointwise as } m \to \infty.$$

By 13.6 we obtain, for any $x \in H$,

$$\begin{aligned} \|\sum_{n=1}^{m} P_nx - x\|^2 &= \|1_{(0,\frac{1}{m+1}]}(B)x\|^2 = (1_{(0,\frac{1}{m+1}]}(B)x|1_{(0,\frac{1}{m+1}]}(B)x) \\ &= (1_{(0,\frac{1}{m+1}]}(B)x|x) \to 0 \ (m \to \infty), \end{aligned}$$

since $1_{(0,\frac{1}{m+1}]} \to 0$ bounded pointwise as $m \to \infty$. Moreover, by orthogonality of the summands we have $\|x\|^2 = \sum_{n\in\mathbb{N}} \|P_nx\|^2$. For $x \in D(A)$ we thus have

$$Ax = \sum_{n\in\mathbb{N}} P_nAx = \sum_{n\in\mathbb{N}} AP_nx \quad \text{and} \quad \sum_{n\in\mathbb{N}} \|AP_nx\|^2 = \|Ax\|^2 < \infty.$$

Now let $x \in H$ such that $\sum_{n\in\mathbb{N}} \|AP_nx\|^2 < \infty$. Then $\sum_{n\in\mathbb{N}} AP_nx$ converges in H, and by (b) we have that

$$x_m := \sum_{n=1}^{m} P_nx \in D(A), \quad Ax_m = \sum_{n=1}^{m} AP_n$$

for each $m \in \mathbb{N}$, and $x_m \to x$ in H, $Ax_m \to y := \sum_{n\in\mathbb{N}} AP_nx$ in H as $m \to \infty$. By closedness of A we obtain $x \in D(A)$ and $Ax = \sum_{n\in\mathbb{N}} AP_nx$.

(d) Let $n \in \mathbb{N}$. Then H_n is invariant under B, since $BP_n = B\theta_n(B) = \theta_n(B)B$ by 13.6. We set $B_n := B|_{H_n}$. Since B commutes with resolvents of A, the space H_n is also invariant under resolvents of A.[5] Hence, for any $\lambda \in \rho(A)$, we have $\lambda \in \rho(A_n)$ and

$$R(\lambda, A_n) = R(\lambda, A)|_{H_n}$$

(in particular, we have $\sigma(A_n) \subseteq \sigma(A)$): Indeed, for $\lambda \in \rho(A)$ the operator $\lambda - A_n$, which is a restriction of $\lambda - A$, is clearly injective. Moreover, for $y \in H_n$ and $x := R(\lambda, A)$ we have $x \in H_n$ and $(\lambda - A_n)x = (\lambda - A)R(\lambda, A)y = y$, and $\lambda - A_n : H_n \to H_n$ is also surjective. Hence we have, for any $n \in \mathbb{N}$,

$$B_n = R(i, A_n)R(-i, A_n), \quad B_n^{-1} = (i - A_n)(-i - A_n) = 1 + A_n^2.$$

Since $\sigma(B_n) \subseteq [\frac{1}{n+1}, \frac{1}{n}]$ by 13.6 we thus have $\sigma((1+A_n^2)^{-1}) \subseteq [\frac{1}{n+1}, \frac{1}{n}]$. By spectral mapping this implies $\sigma(A_n^2) \subseteq [n-1, n]$ and finally

$$\sigma(A_n) \subseteq [-\sqrt{n}, -\sqrt{n-1}] \cup [\sqrt{n-1}, \sqrt{n}],$$

so (d) is proved. □

Remark: But by 13.14 we have

$$\sigma(A) = \overline{\bigcup_{n\in\mathbb{N}} \sigma(A_n)} = \bigcup_{n\in\mathbb{N}} \sigma(A_n)$$

where the last equality is due to the fact that by (d), for any Cauchy sequence in $\bigcup_n \sigma(A_n)$, there exists $n_0 \in \mathbb{N}$ and a convergent subsequence contained in $\sigma(A_{n_0})$. From this we obtain

$$\sigma(A) \cap ((-\sqrt{n}, -\sqrt{n-1}) \cup (\sqrt{n-1}, \sqrt{n})) = \sigma(A_n) \cap ((-\sqrt{n}, -\sqrt{n-1}) \cup (\sqrt{n-1}, \sqrt{n}))$$

for each $n \in \mathbb{N}$.

In 13.14, we have established an orthogonal decomposition of the space H into closed subspaces H_n, $n \in \mathbb{N}$, such that $H_n \subseteq D(A)$, $A_n := A|_{H_n} \in \mathcal{L}(H_n)$ and A_n is self-adjoint in H_n and corresponds to a bounded part of the spectrum of A. We now apply 13.6 to each of the parts A_n and thus define a functional calculus for A for bounded Borel measurable functions on the spectrum $\sigma(A)$.

13.15. Theorem: Let A be a self-adjoint operator in H and, for $n \in \mathbb{N}$, let P_n, H_n, A_n be as in 13.14. For $f \in \mathcal{B}_b(\sigma(A))$ we define

$$f(A)x := \sum_{n\in\mathbb{N}} f(A_n)P_n x, \quad x \in H.$$

[5] The following argument also proves $(*)$ in the proof of (a) above: Let $T \in \mathcal{L}(H)$ such that $TB = BT$. Then $Tp(B) = p(B)T$ for all polynomials p. By Weierstraß, one can approximate continuous functions on $\sigma(B)$ by polynomials, and this gives $Tf(B) = f(B)T$ for all $f \in C(\sigma(B))$. Finally, application of 13.2 shows $Tf(B) = f(B)T$ for all $f \in \mathcal{B}_b(\sigma(B))$, where 13.2(2) is shown with the help of σ-continuity of the map Ψ from 13.6.

Here and in $(*)$ this is applied to $f = \theta_n$ and $T = R(\lambda, A)$, $T = C$, respectively.

Then $f(A) \in \mathcal{L}(H)$ is well-defined, $\|f(A)\| \le \|f\|_\infty$, and the map $\Psi : \mathcal{B}(\sigma(A)) \to \mathcal{L}(H)$ is an algebra homomorphism with $\Psi(1_{\sigma(A)}) = I$, $\Psi((z - (\cdot))^{-1}) = R(z, A)$ for $z \in \mathbb{C} \setminus \mathbb{R}$.

For $f \in \mathcal{B}_b(\sigma(A))$ one has $\Psi(f)^* = \Psi(\overline{f})$. In particular, each $f(A)$ is normal, $f(A)$ is self-adjoint for real-valued f, and $f(A) \ge 0$ if $f \ge 0$.

If $f_m \to f$ bounded pointwise as $m \to \infty$ then $f_m(A)x \to f(A)x$ for any $x \in H$.

Proof. Since $f(A_n) \in \mathcal{L}(H_n)$ with $\|f(A_n)\| \le \|f\|_{\infty,\sigma(A_n)} \le \|f\|_{\infty,\sigma(A)}$ we have

$$\sum_{n\in\mathbb{N}} \|f(A_n)P_n x\|^2 \le \|f\|^2_{\infty,\sigma(A)} \sum_{n\in\mathbb{N}} \|P_n x\|^2 = \|f\|^2_{\infty,\sigma(A)} \|x\|^2, \quad x \in H.$$

Hence $\sum_n f(A_n)P_n x$ converges and

$$\|f(A)x\| = \Big(\sum_{n\in\mathbb{N}} \|f(A_n)P_n x\|^2\Big)^{1/2} \le \|f\|_{\infty,\sigma(A)} \|x\|.$$

This implies that $\Psi : \mathcal{B}_b(\sigma(A)) \to \mathcal{L}(H)$ is linear and continuous with norm ≤ 1. $\Psi(1_{\sigma(A)}) = I_H$ follows from 13.14(c). For $x \in H$ and $z \in \mathbb{C} \setminus \mathbb{R}$ we have (cp. 13.14(d) and its proof)

$$R(z, A)x = \sum_{n\in\mathbb{N}} R(z, A)P_n x = \sum_{n\in\mathbb{N}} R(z, A_n)P_n x.$$

By 12.6 we have $R(z, A_n) = (z - (\cdot))^{-1}(A_n)$ for any $n \in \mathbb{N}$, so $R(z, A) = \Psi((z - (\cdot))^{-1})$ is proved.

Notice that, for each $m \in \mathbb{N}$,

$$P_m f(A)x = \sum_{n\in\mathbb{N}} P_m f(A_n)P_n x = f(A_m)P_m x.$$

Hence, for $f, g \in \mathcal{B}_b(\sigma(A))$ and $x \in H$,

$$(gf)(A) = \sum_{n\in\mathbb{N}} (gf)(A_n)P_n x = \sum_{n\in\mathbb{N}} g(A_n)f(A_n)P_n x = \sum_{n\in\mathbb{N}} g(A_n)P_n f(A)x = g(A)(f(A)x),$$

and Ψ is multiplicative. By orthogonality of the summands we have, for $f \in \mathcal{B}_b(\sigma(A))$ and $x, y \in H$,

$$\begin{aligned}
(f(A)x|y) &= \sum_{n\in\mathbb{N}} (P_n f(A)x|P_n y) = \sum_{n\in\mathbb{N}} (f(A_n)P_n x|P_n y)_{H_n} \\
&= \sum_{n\in\mathbb{N}} (P_n x|f(A_n)^* P_n)_{H_n} = \sum_{n\in\mathbb{N}} (P_n x|\overline{f}(A_n)P_n(y))_{H_n} \\
&= (x|\overline{f}(A)y),
\end{aligned}$$

which implies that $f(A)^* = \overline{f}(A)$.

The assertions on normality, self-adjointness and positivity follow as before. Let $(f_m)_{m\in\mathbb{N}}$ be a sequence in $\mathcal{B}_b(\sigma(A))$ that converges bounded pointwise to a function f. By linearity of Ψ we may assume that $f = 0$. We have for the functional calculus from 13.6 that $f_m(A_n)x \to 0$ as $m \to \infty$ for all $n \in \mathbb{N}$ and $x \in H_n$. Hence, for fixed $x \in H$, we obtain

$$f_m(A_n)P_n x \to 0 \quad \text{for each } n \in \mathbb{N}.$$

Letting $M := \sup_{m\in\mathbb{N}} \|f_m\|_\infty$ we have, for any $N \in \mathbb{N}$,

$$\|f_m(A)x\|^2 = \sum_{n\in\mathbb{N}} \|f_m(A_n)P_n x\|^2 \leq \sum_{n=1}^{N} \|f_m(A_n)P_n x\|^2 + M \sum_{n>N} \|P_n x\|^2.$$

For a given $\varepsilon > 0$ we now choose N such that $M\sum_{n>N} \|P_n x\|^2 \leq \varepsilon/2$, and then m_0 such that $\sum_{n=1}^{N} \|f_m(A)P_n x\|^2 \leq \varepsilon/2$ for all $m \geq m_0$. Thus $f_m(A)x \to 0$ in H is proved. □

Remark: One can show that $\Psi : \mathcal{B}_b(\sigma(A)) \to \mathcal{L}(H)$ with the stated properties is unique. Moreover, one can actually see that $P_n = (\theta_n \circ \psi)(A)$ where $\psi(z) = \frac{1}{1+z^2}$, so $P_n = \eta_n(A)$ where $\eta_n = 1_{(-\sqrt{n},-\sqrt{n-1}]\cup[\sqrt{n-1},\sqrt{n})}$.

We briefly mention another aspect of the Spectral Theorem.

13.16. Spectral measures: Let A be self-adjoint in H and let Ψ denote the functional calculus from 13.15. For all $x, y \in H$

$$\mu_{x,y} : \mathcal{B}(\mathbb{R}) \to \mathbb{C}, \quad M \mapsto \mu_{x,y}(M) := (\Psi(1_M)x|y) = (1_M(A)x|y),$$

is a compex Borel measure on $\mathbb{R}$. For $x = y$, $\mu_{x,x}$ is a positive finite measure with

$$\mu_{x,x}(\mathbb{R}) = (1_{\mathbb{R}}(A)x|x) = (x|x) = \|x\|^2,$$

i.e., for $x \in H$ with $\|x\| = 1$, $\mu_{x,x}$ is a probability maesure on $\mathbb{R}$.
For each $f \in \mathcal{B}_b(\sigma(A))$ we have the repreesentation[6]

$$(f(A)x|y) = \int f(\lambda)\, d\mu_{x,y}(\lambda).$$

One wries this as

$$f(A) = \int f(\lambda)\, dE(\lambda),$$

where the *spectral measure* (or *spectral resolution* E of A is given by

$$E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(H), \quad M \mapsto E(M) := \Psi(1_M) = 1_M(A).$$

[6] If A is bounded this is similar to what we have done in the construction of the functional calculus in 13.6.

Note that E is not σ-additive w.r.t. operator norm. By 13.15 however, we have that $\mathcal{B}(\mathbb{R}) \to H$, $M \mapsto E(M)x$, is σ-additive for each $x \in H$. For $f \in \mathcal{B}_b(\mathbb{R})$ and $x \in H$ we then have

$$\|f(A)x\|^2 = (f(A)x|f(A)x) = (|f|^2(A)x|x) = \int |f|^2\, d(E(\cdot)x|x).$$

This opens a way to define $f(A)$ as (in general) *unbounded* operator in H for each Borel-measurable function $f : \sigma(A) \to \mathbb{C}$ with domain

$$D(f(A)) = \{x \in H : \int |f|^2\, d(E(\cdot)x|x) < \infty\},$$

see, e.g., Rudin: Functional Analysis.

The following is another version of the Spectral Theorem (here we omit the proof, ee, e.g., Davies: Spectral Theory and Differential Operators).

13.17. Spectral Theorem (multiplicator version): Leti A be a self-adjoint operator in the separable Hilbert space H. then there exists a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$, a measurable function $m : \Omega \to \sigma(A)$ and a unitary operator $U : H \to L^2(\mu)$ such that

$$D(A) = \{x \in H : m \cdot Ux \in L^2(\mu)\}, \quad Ax = U^{-1}\big(m \cdot Ux\big).$$

The functional calculus for the operator A is then obtained from the functional calculus for the multiplication operator $f \mapsto m \cdot f$ in $L^2(\mu)$.

We come back to the question of (essential) self-adjointness for symmetric operators. First we present an example.

Example: Let $A = -\Delta$ with $D(A) = C_c^\infty(\mathbb{R}^d)$ in $H = L^2(\mathbb{R}^d)$. Clearly, A is densely defined and symmetric. We determine the adjoint operator A^*. For $f, g \in L^2(\mathbb{R}^d)$ we have

$$\begin{aligned} f \in D(A^*) \text{ and } A^*f = g \iff & \forall\, \varphi \in C_c^\infty(\mathbb{R}^d) : (-\Delta\varphi|f) = (f|g) \\ \iff & \forall\, \varphi \in C_c^\infty(\mathbb{R}^d) : \int 4\pi^2|\xi|^2\hat{\varphi}(\xi)\overline{\hat{f}(\xi)}\, d\xi = \int \hat{\varphi}(\xi)\overline{\hat{g}(\xi)}\, d\xi. \end{aligned}$$

Since $\hat{f}, \hat{g} \in L^2(\mathbb{R}^d)$ we obtain equality of the integrals for all $\hat{\varphi} \in \mathscr{S}(\mathbb{R}^d)$. By $C_c^\infty(\mathbb{R}^d) \subset \mathscr{S}(\mathbb{R}^d)$ and the corollary in FA 5.2 we then obtain

$$\begin{aligned} f \in D(A^*) \text{ and } A^*f = g \iff & 4\pi^2|\xi|^2\hat{f}(\xi) = \hat{g}(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^d \\ \iff & f \in H^2(\mathbb{R}^d) \text{ and } -\Delta f = g, \end{aligned}$$

where for the last equivalence we used $f, g \in L^2(\mathbb{R}^d)$ and the definitions of $H^2(\mathbb{R}^d)$ and the (weak) Laplace operator. Thus we have $A^* = -\Delta$ with $D(A^*) = H^2(\mathbb{R}^d) = W^{2,2}(\mathbb{R}^d)$.

Finally, A is essentially self-adjoint, since the graph norm of A is equivalent to $\|\cdot\|_{W^{2,2}}$ and $D(A) = C_c^\infty(\mathbb{R}^d)$ is dense in $(W^{2,2}(\mathbb{R}^d), \|\cdot\|_{W^{2,2}})$ (see FA 5.4).

For the general case we state the following in the situation of 13.9.

13.18. Lemma: Let A be symmetric and closed. If $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$ such that $\operatorname{Im} \lambda \cdot \operatorname{Im} \mu > 0$ then

$$\operatorname{codim} R(\lambda - A) = \operatorname{codim} R(\mu - A).$$

Proof. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. We show the assertion for $\mu \in \mathbb{C} \setminus \mathbb{R}$ with $|\lambda - \mu| < |\operatorname{Im} \lambda|$. The lemma follows by taking $\lambda = \pm in$ and $n \to \infty$. We let $X := [D(A)]$. By 13.9, $\lambda - A$ is injective and $Y := R(\lambda - A)$ is closed. Let $Z := X \times Y^\perp$ and consider

$$\widehat{z - A} : X \times Y^\perp \to H, \quad (x, y) \mapsto (z - A)x + y, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

For $z = \lambda$, this operator is bijective with inverse $\widehat{R}h := ((\lambda - A)^{-1}Ph, (I - P)h)$, where P denotes the orthogonal projection from H onto Y. If $|\mu - \lambda| < |\operatorname{Im} \lambda|$, then

$$\widehat{\mu - A} = \widehat{\lambda - A} + (\mu - \lambda)\pi_1 = (I_H + (\mu - \lambda)\pi_1 \widehat{R})\widehat{\lambda - A},$$

where $\pi_1 : X \times Y^\perp \to H$, $(x, y) \mapsto x$, and $\|\pi_1 R\| = \|(\lambda - A)^{-1}P\|_{\mathcal{L}(H)} \leq 1/|\operatorname{Im} \lambda|$ by 13.9. Hence

$$(\widehat{\mu - A})^{-1} = \widehat{R} \sum_{k=0}^{\infty} (-\pi_1 \widehat{R})^k (\mu - \lambda)^k$$

and $\widehat{\mu - A}$ is bijective $X \times Y^\perp \to H$. In particular, $R(\mu - A) + Y^\perp = H$. Moreover, if $x \in D(A)$ and $(\mu - A)x = -y \in Y^\perp$ then $\widehat{\mu - A}(x, y) = 0$, so $x = 0 = y$, which implies $R(\mu - A) \cap Y^\perp = \{0\}$. This means that $Y^\perp$ is a complement both for $R(\lambda - A)$ and $R(\mu - A)$, so $\operatorname{codim} R(\lambda - A) = \operatorname{codim} R(\mu - A)$. □

Remark: The proof is similar to what we have done for Fredholm operators. The numbers $n_\pm(A) := \operatorname{codim} R(\pm i - A)$ are called *defect indices*, and it can be shown that A has a self-adjoint extension if and only if $n_+(A) = n_-(A)$.

If A is in addition positive, i.e. $(Ax|x) \geq 0$ for all $x \in D(A)$, then one can show in a similar way that $n_+(A) = n_-(A) = \operatorname{codim}(\mu + A)$ for any $\mu > 0$, and A has self-adjoint extensions. We will present how to get one in the next section. end Tue 09.07.19

14 Functional calculus for sectorial operators

A very nice way to obtain self-adjoint (and more general) operators in a Hilbert space is via sesquilinear forms. We have seen this in 13.5 for the bounded case, but here we want to define unbounded operators. We start with a motivation.

14.1. The Dirichlet form: We recall the *Divergence Theorem*:

> Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\partial\Omega \in C^1$ and $U \supseteq \overline{\Omega}$ be open. Then, for $F \in C^1(U)$,
> $$\int_\Omega \operatorname{div} F \, dx = \int_{\partial\Omega} \nu \cdot F \, d\sigma,$$
> where $\nu : \partial\Omega \to \mathbb{R}^d$ denotes the outer unit normal.

Letting $F = \overline{v}\nabla u$ we obtain

Corollary: Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\partial\Omega \in C^1$ and let $U \supseteq \overline{\Omega}$ be open. Then, for $u \in C^2(U)$ and $v \in C^1(U)$:

$$\int_\Omega \nabla u \cdot \overline{\nabla v} \, dx = -\int_\Omega (\Delta u)\overline{v} \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\overline{v} \, d\sigma. \qquad (*)$$

Now let $\Omega \subseteq \mathbb{R}^d$ be an arbitrary domain. For $u, v \in W^{1,2}(\Omega)$ we define

$$\mathfrak{a}(u,v) := \int_\Omega \nabla u \cdot \overline{\nabla v} \, dx.$$

Then $\mathfrak{a} : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$ is sesquilinear, $\mathfrak{a}(u,v) = \overline{\mathfrak{a}(v,u)}$ for all $u, v \in W^{1,2}(\Omega)$, and

$$\operatorname{Re} \mathfrak{a}(u,u) = \int_\Omega |\nabla u|^2 \, dx \geq 0.$$

Notice that

$$\begin{aligned} (u|v)_{W^{1,2}} &= \int_\Omega \nabla u \cdot \overline{\nabla v} + u\overline{v} \, dx = \mathfrak{a}(u,v) + (u|v)_{L^2} \\ \|u\|^2_{W^{1,2}} &= \int_\Omega |\nabla u|^2 + |u|^2 \, dx = \mathfrak{a}(u,u) + \|u\|^2_{L^2}. \end{aligned}$$

In particular, $\mathfrak{a}(\cdot,\cdot) + (\cdot|\cdot)_{L^2}$ is a scalar product on $W^{1,2}(\Omega)$ and $W^{1,2}(\Omega)$ is complete w.r.t. this scalar product. This also holds for the closed linear subspace

$$W_0^{1,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{1,2}}}$$

of $W^{1,2}(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$, also $W_0^{1,2}(\Omega)$ and $W^{1,2}(\Omega)$ are dense in $L^2(\Omega)$.

We shall define the Laplace operator with Dirichlet and with Neumann boundary conditions in arbitrary domains $\Omega \subseteq \mathbb{R}^d$ via the Dirichlet form $\mathfrak{a}$ with domain $W_0^{1,2}(\Omega)$ and $W^{1,2}(\Omega)$, respectively. We first remark that, for a bounded domain Ω with C^1-boundary and an open superset $U \supseteq \overline{\Omega}$, we have:

- If $u \in C^2(U)$ with $u = 0$ on $\partial\Omega$ and $v \in C_c^\infty(\Omega)$ then

$$\mathfrak{a}(u,v) = \int_\Omega (-\Delta u)\overline{v}\,dx = (-\Delta u|v)_{L^2}.$$

- If $u \in C^2(U)$ with $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ and $v \in C^1(U)$ then

$$\mathfrak{a}(u,v) = \int_\Omega (-\Delta u)\overline{v}\,dx = (-\Delta u|v)_{L^2}.$$

14.2. Sesquilinear forms: Let H be a Hilbert space and $V \subseteq H$ be a dense linear subspace. A sesquilinear form $\mathfrak{a} : V \times V \to \mathbb{C}$ is called

- *symmetric* if $\mathfrak{a}(u,v) = \overline{\mathfrak{a}(v,u)}$ for all $u, v \in V$,
- *accretive*, if $\operatorname{Re}\mathfrak{a}(u,u) \geq 0$ for all $u \in V$,
- *sectorial*, if there exists $\omega \in [0, \pi/2)$ such that $\mathfrak{a}(u,u) \in \Sigma_\omega$ for all $u \in V$, where $\Sigma_0 := [0,\infty)$ and

  $$\Sigma_\omega := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \omega\} \cup \{0\} \qquad \text{for } \omega \in (0, \pi/2).$$

  In this case $\mathfrak{a}$ is called *sectorial of angle* ω.

If $\mathfrak{a}$ is sectorial of angle ω then $\mathfrak{a}$ is accretive and

$$|\operatorname{Im}\mathfrak{a}(u,u)| \leq \tan\omega \, \operatorname{Re}\mathfrak{a}(u,u), \qquad u \in V.$$

The *adjoint form* $\mathfrak{a}^* : V \times V \to \mathbb{C}$ of $\mathfrak{a}$ is defined by $\mathfrak{a}^*(u,v) = \overline{\mathfrak{a}(v,u)}$. We clearly have $(\mathfrak{a}^*)^* = \mathfrak{a}$; $\mathfrak{a} = \mathfrak{a}^* \Leftrightarrow \mathfrak{a}$ is symmetric; $\mathfrak{a}$ is accretive $\Leftrightarrow \mathfrak{a}^*$ is accretive; $\mathfrak{a}$ is sectorial of angle $\omega \Leftrightarrow \mathfrak{a}^*$ is sectorial of angle ω.

For each sesquilinear form $\mathfrak{a}$ the sesquilinear forms

$$\operatorname{Re}\mathfrak{a} := \frac{1}{2}\big(\mathfrak{a} + \mathfrak{a}^*\big), \qquad \operatorname{Im}\mathfrak{a} := \frac{1}{2i}\big(\mathfrak{a} - \mathfrak{a}^*\big)$$

are symmetric and we have $\mathfrak{a} = \operatorname{Re}\mathfrak{a} + i\operatorname{Im}\mathfrak{a}$.

WARNING: We have $(\operatorname{Re}\mathfrak{a})(u,u) = \operatorname{Re}\big(\mathfrak{a}(u,u)\big)$ for all $u \in V$, but $(\operatorname{Re}\mathfrak{a})(u,v)$ is not real in general! The same holds for $\operatorname{Im}\mathfrak{a}$.

Estimates: If $\mathfrak{a}$ is sectorial of angle ω then we have for all $u, v \in V$:

$$\begin{aligned}
|(\operatorname{Re}\mathfrak{a})(u,v)| &\leq \operatorname{Re}\mathfrak{a}(u,u)^{1/2}\operatorname{Re}\mathfrak{a}(v,v)^{1/2},\\
|(\operatorname{Im}\mathfrak{a})(u,v)| &\leq (\tan\omega)\operatorname{Re}\mathfrak{a}(u,u)^{1/2}\operatorname{Re}\mathfrak{a}(v,v)^{1/2},\\
|\mathfrak{a}(u,v)| &\leq (1+\tan\omega)\operatorname{Re}\mathfrak{a}(u,u)^{1/2}\operatorname{Re}\mathfrak{a}(v,v)^{1/2}.
\end{aligned}$$

Proof. The first inequality is Cauchy-Schwarz, and the third inequality is implied by the first and the second.

For the proof of the second inequality we may assume that $(\operatorname{Im} a)(u,v)$ is real (otherwise we multiply u with a suitable $\gamma \in \mathbb{C}$ with $|\gamma| = 1$). Then we have

$$(\operatorname{Im} \mathfrak{a})(u,v) = \frac{1}{4}\big((\operatorname{Im} \mathfrak{a})(u+v,u+v) - (\operatorname{Im} \mathfrak{a})(u-v,u-v)\big),$$

and by sectoriality of $\mathfrak{a}$ thus

$$\begin{aligned} |(\operatorname{Im} \mathfrak{a})(u,v)| &\leq \frac{\tan\omega}{4}\big((\operatorname{Re} \mathfrak{a})(u+v,u+v) + (\operatorname{Re} \mathfrak{a})(u-v,u-v)\big) \\ &= \frac{\tan\omega}{2}\big(\operatorname{Re} \mathfrak{a}(u,u) + \operatorname{Re} \mathfrak{a}(v,v)\big). \end{aligned}$$

For each $\alpha > 0$ we hence have

$$\begin{aligned} |(\operatorname{Im} \mathfrak{a})(u,v)| &= |(\operatorname{Im} \mathfrak{a})(\alpha u, \alpha^{-1} v)| \\ &\leq \frac{\tan\omega}{2}\big(\alpha^2 \operatorname{Re} \mathfrak{a}(u,u) + \alpha^{-2} \operatorname{Re} \mathfrak{a}(v,v)\big). \end{aligned}$$

If $\operatorname{Re} \mathfrak{a}(v,v) \operatorname{Re} \mathfrak{a}(u,u) = 0$ then the assertion follows letting $\alpha \to 0+$ or $\alpha \to \infty$. If $\operatorname{Re} \mathfrak{a}(v,v) \operatorname{Re} \mathfrak{a}(u,u) \neq 0$ then the assertion follows with $\alpha = \big(\operatorname{Re} \mathfrak{a}(v,v) / \operatorname{Re} \mathfrak{a}(u,u)\big)^{1/4}$. $\square$

Remark: If $\mathfrak{a} : V \times V \to \mathbb{C}$ is sectorial of angle ω and $\delta > 0$ then

$$(u|v)_V := (\operatorname{Re} \mathfrak{a})(u,v) + \delta (u|v)_H$$

defines a scalar product on V (usually one takes $\delta = 1$, all these scalar products are equivalent). In this case the sesquilinear form $\mathfrak{a}$ i called *closed (on V)* if V is a Hilbert space w.r.t. $(\cdot|\cdot)_V$. Obviously, $\mathfrak{a}$ is closed if and only if $\mathfrak{a}^*$ is closed.

Example: In 14.1 the Dirichlet form $\mathfrak{a}$ is symmetric, sectorial of angle 0 and closed, both on $V = W^{1,2}(\Omega)$ and on $V = W_0^{1,2}(\Omega)$. Thus we can apply the following proposition in both situations. end Thu 11.07.19

14.3. Proposition: Let H be a Hilbert space and $V \subseteq H$ a dense linear subspace. Let $\mathfrak{a} : V \times V \to \mathbb{C}$ be a sesquilinear which is sectorial of angle ω and closed. Define the linear operator A in H by the following for all $u, f \in H$:

$$u \in D(A) \text{ and } Au = f \;:\Longleftrightarrow\; u \in V \text{ and } \forall v \in V : \mathfrak{a}(u,v) = (f|v).$$

Then A is a closed linear operator in H, $D(A)$ is dense in $(V, (\cdot|\cdot)_V)$, we have $\sigma(A) \subseteq \Sigma_\omega$ and

$$\|R(\lambda, A)\|_{\mathcal{L}(H)} \leq 1/d(\lambda, \Sigma_\omega) \qquad \text{for all } \lambda \in \mathbb{C} \setminus \Sigma_\omega.$$

Moreover, we have $(Ax|x)_H \in \Sigma_\omega$ for all $x \in D(A)$. If we define in this way an operator for the adjoint sesquilinear form $\mathfrak{a}^*$ then we obtain the adjoint operator A^*. In particular, A is self-adjoint in H if $\mathfrak{a}$ is symmetric.

Remark: Note that (by denseness of V in H) the rule above really defines a linear operator A in H. This operator is called *the operator associated with* $\mathfrak{a}$, sometimes denoted by $A \sim \mathfrak{a}$.

Proof. We denote by V^* the *antidual space* of $(V, (\cdot|\cdot)_V)$, i.e. the linear space of all continuous antilinear functionals $\phi : V \to \mathbb{C}$. We study the linear operator

$$\mathcal{B} : V \to V^*, \quad u \mapsto \mathfrak{a}(u, \cdot) + (u|\cdot)_H,$$

which is continuous by the estimates in 9.2. For $u \in V$ with $\|u\|_V = 1$ we have

$$\|\mathcal{B}u\|_{V^*} = \sup_{\|v\|_V=1} |\mathfrak{a}(u, v) + (u|v)_H| \geq |\mathfrak{a}(u, u) + (u|u)_H| \geq \operatorname{Re} \mathfrak{a}(u, u) + \|u\|_H^2 = \|u\|_V^2 = 1,$$

and $\mathcal{B}$ is injective and $R(\mathcal{B})$ is closed in V^*. Moreover, $R(\mathcal{B})$ is dense in V^*, otherwise there were (by reflexivity of V and Hahn-Banach) a $v \in V$ with $\mathcal{B}(u)(v) = 0$ for all $u \in V$, which contradicts the estimate we just proved. Hence $\mathcal{B} : V \to V^*$ is an isomorphism and has a continuous inverse $\mathcal{R} : V^* \to V$ with norm ≤ 1.[7]

We identify H with its antidual space H^*. Since V is dense and continuously embedded in H, the injection $H^* \to V^*$ is continuous. Hence we have continuous embeddings $V \hookrightarrow H = H^* \hookrightarrow V^*$. Here we identify $h \in H$ with $(h|\cdot)_H\Big|_V \in V^*$.

Now we let $R := \mathcal{R}|_H$. Then $R \in \mathcal{L}(H)$ and R is injective (since $\mathcal{R}$ is injective). We define $A := R^{-1} - I_H$. Then A is a closed linear operator in H and $(A + 1)^{-1} = R \in \mathcal{L}(H)$, i.e. $-1 \in \rho(A)$. Moreover, we have for $u, f \in H$:

$$\begin{aligned} u \in D(A),\ Au = f \iff & R(f + u) = u \iff u \in V \text{ and } \mathcal{R}(f + u) = u \\ \iff & u \in V \text{ and } \mathcal{B}u = u + f \\ \iff & u \in V \text{ and } \forall v \in V : \mathfrak{a}(u, v) + (u|v)_H = (u|v)_H + (f|v)_H \\ \iff & u \in V \text{ and } \forall v \in V : \mathfrak{a}(u, v) = (f|v)_H. \end{aligned}$$

$D(A)$ is dense in V: We have $D(A) = R(R) = \mathcal{R}(H)$. Since $\mathcal{R} : V^* \to V$ is an isomorphism, it suffices to show that $H = H^*$ is dense in V^*. But otherwise there would exist $v \in V$ with $v \neq 0$ and $(h|v)_H = 0$ for all $h \in H$, which implies $\|v\|_H = 0$, a contradiction.

Now let the operator B be associated with the sesquilinear form $\mathfrak{a}^*$ via

$$v \in D(B) \text{ and } Bv = g \ :\iff \ v \in V \text{ and } \forall w \in V : \mathfrak{a}^*(v, w) = (g|w).$$

Then we have for $u \in D(A)$ and $v \in D(B)$: $u, v\|V\|$ and

$$(Au|v) = \mathfrak{a}(u, v) = \overline{\mathfrak{a}^*(v, u)} = \overline{(Bv|u)} = (u|Bv).$$

This implies $v \in D(A^*)$ and $A^*v = Bv$, i.e. $B \subseteq A^*$. By $-1 \in \rho(B)$ and $-1 \in \rho(A^*)$ (which follows from $-1 \in \rho(A)$) we obtain $B = A^*$. If $\mathfrak{a}$ is symmetric then $\mathfrak{a}^* = \mathfrak{a}$ and we obtain $A = A^*$, i.e. A is self-adjoint.

[7] This also follows from Lax-Milgram.

For all $u \in D(A)$ we obviously have $(Au|u)_H = \mathfrak{a}(u,u) \in \Sigma_\omega$. For $\lambda \in \mathbb{C} \setminus \Sigma_\omega$ and $u \in D(A)$ with $\|u\| = 1$ we thus have

$$\|(\lambda - A)u\| \geq |((\lambda - A)u|u)| = |\lambda - (Au|u)| \geq d(\lambda, \Sigma_\omega).$$

Hence $\lambda - A$ is injective and $R(\lambda - A)$ is closed in H. But then also $(\lambda - A)^* = \overline{\lambda} - A^*$ is injective, which implies that $R(\lambda - A)$ is dense in H. We conclude $\lambda \in \rho(A)$ and $\|R(\lambda, A)\| \leq 1/d(\lambda, \Sigma_\omega)$ as asserted. □

Examples: (1) Let $\Omega \subseteq \mathbb{R}^d$ be an arbitrary domain. Then the Dirichlet Laplace operator $-\Delta_D$ on Ω is the operator associated with the Dirichlet form $\mathfrak{a}$ from 14.1 on $V_D = W_0^{1,2}(\Omega)$, and the Neumann Laplace operator $-\Delta_N$ on Ω is the operator assciated with the Dirichlet form $\mathfrak{a}$ from 14.1 on $V_N = W^{1,2}(\Omega)$. Note that we have $V_D \subseteq V_N$ but $D(-\Delta_D) \not\subseteq D(-\Delta_N)$ in general, since then $-1 \in \rho(-\Delta_D) \cap \rho(-\Delta_N)$ implied $-\Delta_D = -\Delta_N$ (which is false, e.g., for bounded domains with C^1-boundary). end Tue 16.07.19

(2) Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\partial\Omega \in C^1$ and let $a : \Omega \to \mathbb{C}^{d \times d}$ be bounded and measurable such that $\overline{\xi}^t a(x)\xi \in \Sigma_\omega$ for a.e. $x \in \Omega$, all $\xi \in \mathbb{C}^d$ and some $\omega \in [0, \pi/2)$. Assume there exists $\eta > 0$ such that

$$\operatorname{Re} \overline{\xi}^t a(x)\xi \geq \eta|\xi|^2 \qquad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{C}^d.$$

Then the sesquilinear form

$$\mathfrak{a}(u,v) = \int_\Omega (a(x)\nabla u) \cdot \overline{\nabla v}\, dx, \qquad u, v \in V = W^{1,2}(\Omega),$$

is sectorial of angle ω and closed in $H = L^2(\Omega)$. The operator associated with $\mathfrak{a}$ is formally given by $Au = -\operatorname{div}(a(\cdot)\nabla u)$ with boundary conditions $\nu \cdot a(\cdot)\nabla u = 0$ on $\partial\Omega$ (*conormal boundary condition*).

14.4. Closable forms: Let H be a Hilbert space and $V \subseteq H$ be a dense linear subspace. Let $\mathfrak{a} : V \times V \to \mathbb{C}$ be a sectorial form that is not closed. We can extend $\mathfrak{a}$ uniquely to a continuous sesquilinear form $\widetilde{\mathfrak{a}}$ on the completion $\widetilde{V}$ of $(V, (\cdot|\cdot)_V)$. This extended form is again sectorial of the ame angle. In order to apply Proposition 9.3 we have to realize $\widetilde{V}$ as a linear subspace of H. This is pssible if and only if the following holds:

For each $\|\cdot\|_V$-Cauchy sequence (u_n) in V with $\|u_n\|_H \to 0$ one has $\|u_n\|_V \to 0$.

Sectorial forms satisfying this property are called *closable*. The condition is equivalent to the continuous extension of the embedding $J : V \to H$, $u \mapsto u$, onto $\widetilde{V}$ being *injective*. In this case, $\widetilde{\mathfrak{a}}$ is clled the *closure of* $\mathfrak{a}$.

Examples: (1) Let A be densely defined and symmetric in H with $(Ax|x) \geq 0$ for all $x \in D(A)$. Then $\mathfrak{a}(u,v) := (Au|v)$ defines a symmetric sesquilinear form on $V := D(A)$

which is sectorial of angle 0. This form is closable (see exercise) and the closure of this form is associated with the so-called *Friedrichs extension* $\widetilde{A}$ of A. Note that $\widetilde{A}$ is self-adjoint in H with $(\widetilde{A}u|u) \geq 0$ for all $u \in D(\widetilde{A})$.

(2) Let $H = L^2(0,1)$, $V = C[0,1]$ and $\mathfrak{a} : V \times V \to \mathbb{C}$ given by $\mathfrak{a}(u,v) = u(0)\overline{v(0)}$. Then $\mathfrak{a}$ is symmetric and sectorial, but not closed. Consider, e.g., $u_n(t) = (1-nt)1_{[0,1/n]}(t)$. Then (u_n) is Cauchy w.r.t. $\|u\|_V = \sqrt{|u(0)|^2 + \|u\|_{L^2}^2}$ with $\|u_n\|_{L^2} \to 0$, but $\|u_n\|_V \to 1$.

14.5. Square roots: Let A be self-adjoint in H with $(Ax|x) \geq 0$ for all $x \in D(A)$ and let P_n, H_n, and A_n be as in 13.14. We define the operator $A^{1/2}$ via

$$\begin{aligned} D(A^{1/2}) &= \{x \in H : \sum_n \|A_n^{1/2} P_n x\|^2 < \infty\}, \\ A^{1/2} x &= \sum_n A_n^{1/2} P_n x \quad \text{for } x \in D(A^{1/2}). \end{aligned}$$

Then $D(A) \subseteq D(A^{1/2})$ and $D(A)$ is dense in $D(A^{1/2})$ w.r.t. the graph norm of $A^{1/2}$. Moreover, $A^{1/2}$ is self-adjoint in H, $A^{1/2} \geq 0$, and $A^{1/2}A^{1/2} = A$, i.e.

$$D(A) = \{x \in D(A^{1/2}) : A^{1/2}x \in D(A^{1/2})\}, \qquad Ax = A^{1/2}A^{1/2}x \text{ for } x \in D(A).$$

Sketch of proof. [8] For the operators $A_n \in \mathcal{L}(H_n)$, we use 13.6, in particular $A_n^{1/2}$ is self-adjoint in H_n and $A_n^{1/2}A_n^{1/2} = A_n$. For $x \in D(A)$ we have by Cauchy-Schwarz

$$\sum_n \|A_n^{1/2}P_n x\|^2 = \sum_n (A_n P_n | P_n x) \leq \Big(\sum_n \|A_n P_n x\|^2\Big)^{1/2} \Big(\sum_n \|P_n x\|^2\Big)^{1/2} = \|Ax\|\|x\|,$$

hence we have $D(A) \subseteq D(A^{1/2})$. The set of all $x \in H$ with $P_n x \neq 0$ just for finitely many n is a subset of $D(A)$ and dense in $D(A^{1/2})$ w.r.t. to the graph norm of $A^{1/2}$. Symmetry of $A^{1/2}$ is clear. By $-1 - A = (i \pm A^{1/2})(i \mp A^{1/2})$ the operator $i \pm A^{1/2}$ is surjective. This implies self-adjointness by 13.9 und 13.11. Checking the algebraic properties is easy (see also the mentioned exercise). □

Now let V be dense in H and $\mathfrak{a} : V \times V \to \mathbb{C}$ be symmetric, sectorial and closed, and $A \sim \mathfrak{a}$. Then

$$D(A^{1/2}) = V \quad \text{and} \quad \|A^{1/2}u\|_H = \mathfrak{a}(u,u) \text{ for } u \in V.$$

Proof. For $u \in D(A) \subseteq D(A^{1/2}) \cap V$ we have

$$\mathfrak{a}(u,u) = (Au|u) = (A^{1/2}u|A^{1/2}u) = \|A^{1/2}u\|_H^2,$$

and $D(A)$ is dense both in $\big(D(A^{1/2}), \big(\|u\|_H^2 + \|A^{1/2}u\|_H^2\big)^{1/2}\big)$ and (by 14.3) in $\big(V, \big(\mathfrak{a}(u,u) + \|u\|_H^2\big)^{1/2}\big)$. This implies the assertion. □

[8] In an exercise, $f(A)$ had been constructed more generally for Borel measurble and locally bounded $f : \sigma(A) \to \mathbb{C}$. Here, $f = \sqrt{\cdot}$ is locally bounded on $\sigma(A) \subseteq [0,\infty)$.

Square Root Problem (Kato '61/'62): Let V be dense in H, let $\mathfrak{a} : V \times V \to \mathbb{C}$ be sectorial and closed, and $A \sim \mathfrak{a}$, $B \sim \operatorname{Re} \mathfrak{a}$. Do we have $D(B^{1/2}) = D(A^{1/2})$ and $\|B^{1/2}u\| \sim \|A^{1/2}u\|$? Here, $B^{1/2}$ is defined via the holomorphic functional calculus (see below).

The answer is "No", in general (McIntosh). The answer is "Yes" for elliptic operators in divergence form $A = -\operatorname{div}(a(\cdot)\nabla)$ on $\mathbb{R}^d$ (McIntosh et al. '01).

14.6. Operatos with compact resolvents: Let V be dense in H, let $\mathfrak{a} : V \times V \to \mathbb{C}$ be symmetric, sectorial and closed, and $A \sim \mathfrak{a}$. The following are equivalent

(i) A has compact resolvents,

(ii) the embedding $V \hookrightarrow H$ is compact,

(iii) $A^{1/2}$ has compact resolvents.

In this case we can order the eigenvalues of A (with multiplicity) by size $0 \le \lambda_1 \le \lambda_2 \le \dots$ and we have the following "mini-max" formula for each $n \in \mathbb{N}$:

$$\lambda_n = \inf\{\lambda(L) : L \subseteq V, \dim L = n\},$$

where $\lambda(L) = \max\{\mathfrak{a}(u,u) : u \in L, \|u\| = 1\}$.

end Thu 18.07.19

Proof. We find a corresponding orthonormal sequence (e_n) of eigenvectors. Then $Au = \sum_j \lambda_j (u|e_j)e_j$, $A^{1/2}u = \sum_j \lambda_j^{1/2}(u|e_j)e_j$, and $\mathfrak{a}(u,u) = \sum_j \lambda_j |(u|e_j)|^2$. We clearly have $\lambda_1 = \inf\{\mathfrak{a}(u,u) : u \in V\}$. Now let $n \ge 2$ and put $M_n := \operatorname{lin}\{e_1, \dots, e_n\}$. Then $\lambda(M_n) = \lambda_n$. Let, on the other hand, let L be a linear subspace of V of dimension n and let $Pu := \sum_{j=1}^{n-1}(u|e_j)e_j$ be the orthogonal projection in H onto M_{n-1}. Since M_n has dimension $> n-1$, we find $u_0 \in L$ with $\|u_0\| = 1$ and $Pu_0 = 0$, i.e. with $u_0 \perp e_j$ for $j = 1, \dots, n-1$. We then have

$$\mathfrak{a}(u_0,u_0) = \sum_{j=n}^{\infty} \lambda_j |(u_0|e_j)|^2 \ge \lambda_n \sum_{j=n}^{\infty} |(u_0|e_j)|^2 = \lambda_n \|u_0\|^2 = \lambda_n.$$

The assertion is proved. □

More can be found in Section 4.5 in Davies: Spectral Theory and Differential Operators.

Examples: (1) Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain, then the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact (see FA). Hence the formula applies to the Dirichlet Laplace operator on Ω.

(2) Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\partial\Omega \in C^1$, then the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Hence the formula applies to the Neumann Laplace operator on Ω.

14.7. Sectorial operators: Let X be a Banach space and $\omega \in [0, \pi)$. A linear operator A in X is called *sectorial of angle* ω if $\sigma(A) \subseteq \Sigma_\omega$ and for each $\theta \in (\omega, \pi)$ there exists $M_\theta > 0$ such that

$$\|R(\lambda, A)\| \leq \frac{M_\theta}{|\lambda|} \quad \text{for all } \lambda \in \mathbb{C} \setminus \Sigma_\theta.$$

Warning: The definiion of this notion is not uniform in the literature!

Example: Let V be dense in the Hilbert space H, let $\mathfrak{a} : V \times V \to \mathbb{C}$ be sectorial of angle $\omega \in [0, \pi/2)$ and closed, and let $A \sim \mathfrak{a}$. Then A is sectorial of angle ω. In particular, a self-adjoint operator A with $A \geq 0$ is sectorial of angle 0.

14.8. Holomorphic functional calculus for sectorial operators: Let $\omega \in [0, \pi)$, A be sectorial of angle ω in the Banach space X.

Remark: If $D(A)$ is dense in X then, for all $x \in X$, we have $\lambda(\lambda + A)^{-1}x \to x$ as $\lambda \to \infty$. If $R(A)$ is dense in X then, for all $x \in X$ we have $A(\lambda + A)^{-1}x \to x$ as $\lambda \to 0+$ and A is injective. Here, $\lambda > 0$ is real.

Proof. We have $M := \sup_{\lambda>0} \|\lambda(\lambda + A)^{-1}\| < \infty$ and $\sup_{\lambda>0} \|A(\lambda + A)^{-1}\| \leq M + 1$. Hence it is sufficient to show convergence on a dense subspace. For $x \in D(A)$ we have

$$\lambda(\lambda + A)^{-1}x - x = -A(\lambda + A)^{-1}x = -\lambda^{-1} \underbrace{\lambda(\lambda + A)^{-1}Ax}_{\text{bounded}} \to 0 \quad (\lambda \to \infty),$$

and for $x = Ay \in R(A)$ we have

$$A(\lambda + A)^{-1}x - x = -\lambda(\lambda + A)^{-1}x = -\lambda \underbrace{A(\lambda + A)^{-1}y}_{\text{bounded}} \to 0 \quad (\lambda \to 0+).$$

The assertions are proved. □

Assumption: From now on we suppose that A has dense domain and range.

For $\theta \in (0, \pi)$ let Σ_θ^0 denote the interior of Σ_θ, and let $H^\infty(\Sigma_\theta^0)$ denote the space of all holomorphic functions $\varphi : \Sigma_\theta^0 \to \mathbb{C}$ such that there exist $\varepsilon, C > 0$ with $|\varphi(z)| \leq C \min\{|z|^\varepsilon, |z|^{-\varepsilon}\}$ for all $z \in \Sigma_\theta^0$.

Example: For each $\theta \in (0, \pi)$ we have $\varphi(z) = \frac{z}{(1+z)^2} = \frac{1}{1+z} - \frac{1}{(1+z)^2} \in H^\infty(\Sigma_\theta^0)$.

For $\theta \in (\omega, \pi)$ and $\varphi \in H^\infty(\Sigma_\theta^0)$ we define

$$\varphi(A) := \frac{1}{2\pi i} \int_{\Gamma_\sigma} \varphi(\lambda) R(\lambda, A) \, d\lambda,$$

where $\Gamma_\sigma = \partial\Sigma_\sigma$ with $\sigma \in (\omega, \theta)$ is parametrized by $\gamma_\sigma(t) = |t|e^{-i\sigma \operatorname{sgn}(t)}$, $t \in \mathbb{R}$, and the integral is an absolutely convergent improper Riemannian integral with values in $\mathcal{L}(X)$ (or a Bochner integral). Notice that, for λ on Γ_σ,

$$\|\varphi(\lambda)R(\lambda, A)\| \leq CM_\sigma |\lambda|^{-1} \min\{|\lambda|^\varepsilon, |\lambda|^{-\varepsilon}\} = CM_\sigma \min\{|\lambda|^{\varepsilon-1}, |\lambda|^{-1-\varepsilon}\}.$$

By Cauchy's integral theorem the definition does not depend on $\sigma \in (\omega, \theta)$: intersect, for $\omega < \sigma < \widetilde{\sigma} < \theta$ and $0 < r < R$, the set $\Sigma_{\widetilde{\sigma}} \setminus \Sigma_\sigma$ with $\overline{B}(0,R) \setminus B(0,r)$ and integrate $\varphi(\lambda)R(\lambda, A)$ along the boundary. By Cauchy this integral is $= 0$. The estimates of the integrand on $|\lambda| = R$ and $|\lambda| = r$ are $\lesssim r^{\varepsilon-1}$ and $\lesssim R^{-\varepsilon-1}$, respectively, and the length of the intervals are $\sim r$ and $\sim R$, respectively. The integrals over these parts of the boundary thus tend to 0 as $r \to 0+$ and $R \to \infty$, respectively.

The map $H_0^\infty(\Sigma_\theta^0) \to \mathcal{L}(X)$, $\varphi \mapsto \varphi(A)$, is linear and multiplicative. The proof of multiplicativity is very similar to what we have done in 9.1(a), see also 9.3. We only have to use that by Cauchy, for any $g \in H_0^\infty(\Sigma_\theta^0)$,

$$\frac{1}{2\pi i}\int_{\Gamma_\sigma} \frac{g(\lambda)}{\lambda - \mu}\, d\mu = \begin{cases} g(\lambda) & , \lambda \in \Sigma_\sigma^0 \\ 0 & , \lambda \notin \Sigma_\sigma \end{cases} .$$

Clearly, $\varphi(A)$ commutes with resolvents of A. end Tue 23.07.19

A first extension: For

$$F \in H_0^\infty(\Sigma_\theta^0) + \operatorname{span}\Big\{\frac{1}{1+z}, 1\Big\} =: \mathcal{E}(\Sigma_\theta^0),$$

i.e. for $F(z) = \varphi(z) + \frac{a}{1+z} + b$ with $\varphi \in H_0^\infty(\Sigma_\theta^0)$, we define

$$F(A) := \varphi(A) + a(1+A)^{-1} + bI \in \mathcal{L}(X).$$

Then $\mathcal{E}(\Sigma_\theta^0) \to \mathcal{L}(X)$ is linear and multiplicative. Linearity is clear. In order to prove multiplicativity let $G = \psi + \frac{c}{1+z} + d \in \mathcal{E}(\Sigma_\theta^0)$. We study the product with the function F from above. There are nine terms but we only have to look closer at the following three: $\frac{a\psi}{1+z}$, $\frac{c\varphi}{1+z}$, and $\frac{ac}{(1+z)^2}$.

Lemma 1: For $\varphi \in \mathcal{E}(\Sigma_\theta^0)$ we have

$$(1+A)^{-1}\varphi(A) = \frac{1}{2\pi i}\int_{\Gamma_\sigma} \frac{\varphi(\lambda)}{1+\lambda} R(\lambda, A)\, d\lambda.$$

For the proof recall $(1+A)^{-1}R(\lambda, A) = (1+\lambda)^{-1}(R(\lambda, A) - R(-1, A))$ and apply Cauchy's Theorem for sectors.

Lemma 2: If $F \in H_0^\infty(\Sigma_\theta^0) + \operatorname{span}\{(1+z)^{-1}\}$ is holomorphic on a neighborhood of $\overline{B}(0, \delta)$ then

$$F(A) = \frac{1}{2\pi i}\int_{\Gamma_{\sigma,\delta}} F(\lambda)R(\lambda, A)\, d\lambda,$$

where $\Gamma_{\sigma,\delta} = \partial(\Gamma_\sigma \cup B(0,\delta))$ is parametrized by arc length and Γ_σ lies to the left. For $\delta < 1$ we have furthermore

$$(1+A)^{-1}F(A) = \frac{1}{2\pi i}\int_{\Gamma_{\sigma,\delta}} \frac{F(\lambda)}{1+\lambda} R(\lambda, A)\, d\lambda.$$

Proof. By Cauchy the assertion is clear for $\varphi \in H_0^\infty$. So we just have to show it for $F(z) = (1+z)^{-1}$, i.e. we have to show

$$(1+A)^{-1} = \frac{1}{2\pi i} \int_{\Gamma_{\sigma,\delta}} \frac{R(\lambda, A)}{1+\lambda} \, d\lambda.$$

We integrate, for $R > 1$, along the boundary of $(\mathbb{C} \setminus (\Sigma_\theta \cup B(0,\delta)) \cap B(0,R)$. By Cauchy this inegral equals $-R(-1,A) = (1+A)^{-1}$. For $|\lambda| = R$ the integrand satisfies a bound $\lesssim R^{-2}$, where the length of the interval is $\sim R$. The integral over this part of the boundary is therefore $\lesssim R^{-1}$ and vanishes for $R \to \infty$. The proof of the second formula is similar to the proof of Lemma 1. □

Via Lemma 1 and Lemma 2 we obtain multiplicativity. We also obtain for $\varphi(z) = z(1+z)^{-2} = (1+z)^{-1} - (1+z)^{-2}$ that $\varphi(A) = A(1+A)^{-2}$. This operator is injective.

Example: Suppose that $\theta < \pi/2$. For $t > 0$ we have $F(z) = e^{-tz} \in H_0^\infty(\Sigma_\theta^0) + \operatorname{span}\{(1+z)^{-1}\}$. We obtain $e^{-tA}e^{-sA} = e^{-(t+s)A}$ for all $t > 0$. Taking $\delta = 1/t$ we obtain the norm estimate

$$\|e^{-tA}\| \leq \frac{M_\sigma}{\pi} \int_{1/t}^\infty e^{-\cos(\sigma)tr} \frac{dr}{r} + M_\sigma t \int_0^{2\pi} e^{-\mathrm{Re}(te^{i\alpha}/t)}/t \, d\alpha,$$

where the right hand side is independent of t. Hence we have $\sup_{t>0} \|e^{-tA}\| < \infty$. One can show that $z \mapsto e^{-zA}$ is analytic on a suitable sector depending on θ ($\to$ bounded analytic semigroups, evolution equations).

Moreover, one can obtain for each $\varphi \in H_0^\infty(\Sigma_\theta^0)$ by similar techniques $\sup_{t>0} \|\varphi(tA)\| < \infty$. This applies (in case $\theta < \pi/2$) to functions $\varphi(z) = z^k e^{-z}$ with $k \in \mathbb{N}$.

We shall extend the holomorphic functional calculus to certain functions F outside $\mathcal{E}(\Sigma_\theta^0)$, but we allow $F(A)$ to be unbounded. We follow an algebraic approach (cp. M. Haase: The Fuctional Calculus for Sectorial Operators).

14.9. A functional calculus of unbounded operators: In the situation of 14.8 let $F : \Sigma_\theta^0 \to \mathbb{C}$ be a holomorphic function such that there exists $g \in H_0^\infty(\Sigma_\theta^0)$ with $Fg \in H_0^\infty(\Sigma_\theta^0)$ and $g(A)$ injective. Such a g is called a *regularizer for* F. We then define

$$F(A) := g(A)^{-1}(Fg)(A),$$

i.e. for $x, y \in X$ we have $x \in D(F(A))$ and $F(A)x = y$ if and only if $(Fg)(A)x = g(A)y$. Note that $F(A)$ is a closed linear operator.

Example: If there exists $C > 0$ such that $|F(z)| \leq C \max\{|z|^{-n}, |z|^n\}$, then $z \mapsto z^m(1+z)^{-2m}$ is a regularizer for F for every $m > n$.

$F(A)$ is well-defined: Let h be another regularizer for F. Then also gh is a regularizer for F and $(Fg)(A)x = g(A)y$ is equivalent to $h(A)(Fg)(A)x = h(A)g(A)y$. On the left hand side we have $(Fgh)(A)x = g(A)(Fh)(A)x$ and on the right ahnd side we have $g(A)h(A)y$. Hence the condition is equivalent to $(Fh)(A)x = h(A)y$.

$F \mapsto F(A)$ is "linear": If g is a regularizer for F and h is a regularizer for G then gh is a regularizer for F and for G, and linearity is easily shown.

$F \mapsto F(A)$ is "multiplicative": We have $F(A)G(A) \subseteq (FG)(A)$ with $D(F(A)G(A)) = D(G(A)) \cap D((FG)(A))$.

Proof. Let g be a regularizer for F and h be a regularizer for G. Then gh is a regularizer for FG. First let $x \in D(G(A))$ with $G(A)x = y \in D(F(A))$ and $F(A)y = z$. Then

$$(FGgh)(A)x = (Fg)(A)(Gh)(A)x = (Fg)(A)h(A)y = h(A)(Fg)(A)y = h(A)g(A)z = (gh)(A)z,$$

hence $x \in D((FG)(A))$ and $(FG)(A)x = z$. Now let $x \in D(G(A)) \cap D((FG)(A))$ and $y = G(A)x$, $z = (FG)(A)x$. We have to show $y \in D(F(A))$ and $F(A)y = z$. But gh is a regularizer for F and

$$(gh)(A)z = (FGgh)(A)x = (Fg)(A)(Gh)(A)x = (Fg)(A)h(A)y = (Fgh)(A)y,$$

which implies the assertion. □

14.10. Fractional powers: For $\alpha \in \mathbb{R}$ we let $F_\alpha(z) := z^\alpha$. Then F_α has regularizers and $A^\alpha := F_\alpha(A)$ is well-defined. For all $\alpha, \beta \in \mathbb{R}$ we then have

$$A^\alpha A^\beta = A^{\alpha+\beta} \quad \text{on} \quad D(A^\alpha A^\beta) = D(A^\beta) \cap D(A^{\alpha+\beta}).$$

14.11. Bounded H^∞-calculus: Each $F \in H^\infty(\Sigma_\theta^0)$ has regularisizers and $F(A)$ is a well-defined closed operator in X. One says that A has a *bounded $H^\infty(\Sigma_\theta^0)$-calculus* if there exists $C > 0$ such that

$$\|F(A)\| \leq C\|F\|_{\infty,\Sigma_\theta^0} \quad \text{for all } F \in H^\infty(\Sigma_\theta^0).$$

One can show that this is the case if and only if there exists $C > 0$ such that

$$\|\varphi(A)\| \leq \|\varphi\|_{\infty,\Sigma_\theta^0} \quad \text{for all } \varphi \in H_0^\infty(\Sigma_\theta^0).$$

The reason behind is the so-called "convergence lemma". More details on this and the bounded H^∞-functional calculus can be found in Chapter 5 of

M. Haase: The Functional Calculus for Sectorial Operators, Birkhäuser 2006,

and Section 9 of

P. Kunstmann, L. Weis: Maximal L^p-Regularity for Parabolic Equations, Fourier Multiplier Theorems and H^∞-Functional Calculus, in Functional Analytic Methods for Evolution Equations, Springer Lecture Notes 1855, Springer 2004, pp. 65–312.