1.2 Microstructures as energy minimizers

Example 1: Consider the problem:

$$\text{Minimize} \qquad \int_0^1 (u_x^2 - 1)^2 \, dx$$

subject to

$$u(0) = u(1) = 0.$$

The minimum is attained but the set of minimizers is highly degenerate. Every Lipschitz function whose slopes are ± 1 almost everywhere and that attains the boundary values is a minimizer. In particular the weak$*$ closure in $W^{1,\infty}$ of the set of minimizers consists of all functions with Lipschitz constant less then or equal to one that are bounded by $\pm \min(x, 1-x)$.

Example 2 (Bolza, L.C. Young): Consider the problem:

$$\text{Minimize} \qquad I(u) := \int_0^1 (u_x^2 - 1)^2 + u^2 \, dx$$

subject to

$$u(0) = u(1) = 0.$$

The infimum of the functional is zero since there exist rapidly oscillating functions with slope ± 1 whose supremum is arbitrarily small. Indeed if s denotes the periodic extension of the sawtooth function

$$s(x) = \begin{cases} x & \text{on } [0, 1/4) \\ 1/2 - x & \text{on } [1/4, 3/4) \\ x - 1 & \text{on } [3/4, 1) \end{cases} \tag{1.1}$$

then $u_j(x) := j^{-1} s(jx)$ satisfy $I(u_j) \to 0$ as $j \to \infty$. The infimum cannot be attained since there is no function that satisfies simultaneously $u \equiv 0$ and $u_x = \pm 1$ almost everywhere. Minimizing sequences must oscillate and converge weakly (in the Sobolev space $W^{1,4}(0,1)$), but not strongly, to zero.