P.L. LIONS "THE CONCENTRATION-COMPACTNESS PRINCIPLE IN THE CALCULUS OF VARIATIONS. THE LOCALLY COMPACT CASE, PART 1" ANNALES DE L'I.H.P., (1984)

I.3. The concentration-compactness lemma.

In this section, we show heuristically the fact that (S.1)-(S.2) insure the compactness of minimizing sequences. As we just said the argument we give below is heuristic but nevertheless, conveniently adapted and justified in all examples in sections below, will be the key argument that we will always use in the following sections.

The argument is based upon the following lemma, which admits many variants all obtained via similar proofs:

LEMMA I.1. — *Let $(\rho_n)_{n\geq 1}$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying:*

$$\rho_n \geq 0 \text{ in } \mathbb{R}^N, \qquad \int_{\mathbb{R}^N} \rho_n dx = \lambda \tag{6}$$

where $\lambda > 0$ is fixed. Then there exists a subsequence $(\rho_{n_k})_{k\geq 1}$ satisfying one the three following possibilities:

i) (compactness) there exists $y_k \in \mathbb{R}^N$ such that $\rho_{n_k}(\cdot + y_k)$ is tight i. e.*:*

$$\forall \varepsilon > 0, \exists R < \infty, \quad \int_{y_k + B_R} \rho_{n_k}(x)dx \geq \lambda - \varepsilon\,; \tag{7}$$

ii) (vanishing) $\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{y+B_R} \rho_{n_k}(x)dx = 0$, *for all* $R < \infty$;

iii) (dichotomy) there exists $\alpha \in\,]0, \lambda[$ such that for all $\varepsilon > 0$, there exist $k_0 \geq 1$ and $\rho_k^1, \rho_k^2 \in L^1_+(\mathbb{R}^N)$ satisfying for $k \geq k_0$:

$$\begin{cases} \| \rho_{n_k} - (\rho_k^1 + \rho_k^2) \|_{L^1} \leq \varepsilon, \left| \int_{\mathbb{R}^N} \rho_k^1 dx - \alpha \right| \leq \varepsilon, \left| \int_{\mathbb{R}^N} \rho_k^2 dx - (\lambda - \alpha) \right| \leq \varepsilon \\ \text{dist}\,(\text{Supp}\,\rho_k^1, \text{Supp}\,\rho_k^2) \underset{k}{\to} +\infty\,. \end{cases} \tag{8}$$

Let us first explain how we use Lemma I.1 and we will then prove Lemma I.1. We consider first the case when e, j depend on x and we assume that (S.1) holds, and we take a minimizing sequences $(u_n)_{n\geq 1}$ of (M_λ):

$$\mathscr{E}(u_n) \underset{n}{\to} I_\lambda, \qquad J(u_n) = \lambda\,.$$

We apply Lemma I.1 with $\rho_n = j(x, Bu_n(x))$: we find a subsequence $(n_k)_{k\geq 1}$ such that (*i*), (*ii*) or (*iii*) holds for all $k \geq 1$. It is easy to see that (*ii*) cannot occur since we have in view of (S.1): $I_\lambda < I_\lambda^\infty$ and $J(u_n) = \lambda$. Next, if (*iii*) occurs we split u_n exactly as we split ρ_{n_k} (see the proof of Lemma I.1) and find, for all $\varepsilon > 0$, u_k^1, u_k^2 in H satisfying for k large enough: $u_{n_k} = u_k^1 + u_k^2 + v_k$

$$\begin{cases} | J(u_k^1) - \alpha | \leq \varepsilon, \quad | J(u_k^2) - (\lambda - \alpha) | \leq \varepsilon\,; \\ \text{dist}\,(\text{Supp}\,u_k^1, \text{Supp}\,u_k^2) \underset{k}{\to} \infty, \quad \| v_k \| \leq \varepsilon\,. \end{cases}$$