<u>Remark</u>. i) The Sobolev constant S is never attained on a domain $\Omega \subsetneqq \mathbb{R}^n$, $n \geq 3$.

If we suppose that for some such domain Ω there exists $0 < u \in H^1_0(\Omega)$ with $S\|u\|^2_{L^{2^*}} = \|\nabla u\|^2_{L^2}$ a suitable multiple $\tilde{u} = \alpha u$, $\alpha > 0$, solves

$$-\Delta \tilde{u} = \tilde{u}^{2^*-1} \quad \text{in } \Omega,$$
$$\tilde{u} = 0 \quad \text{on } \partial\Omega.$$

From this fact, and from the fact that by scale-invariance $S(\Omega) = S(\mathbb{R}^n) = S$ is independent of the domain we derive a contradiction, as follows.

<u>Case 1</u>. If Ω is bounded, $\Omega \subset B_R(0)$ for some $R > 0$, extending $u(x) = 0$ for $x \in B_R(0) \setminus \Omega$, from u we obtain a minimizer $0 \neq u \in H^1_0(B_R(0))$ of Sobolev's ratio, which is impossible by Pohozaev's theorem 1.4.2.

<u>Case 2</u>. If Ω is unbounded, $\Omega \neq \mathbb{R}^n$, by extending $u(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$ we obtain a $0 \leq \tilde{u} \neq 0$ solving $-\Delta \tilde{u} = \tilde{u}^{2^*-1}$ in $\mathbb{R}^n$ vanishing outside Ω, and the strong maximum principle then gives a contradiction.

ii) On $\mathbb{R}^n$, $n \geq 3$, the Sobolev constant is attained by the Talenti functions

$$u(x) = \left(\frac{1}{1+|x|^2} \right)^{\frac{n-2}{2}}$$

and their rescalings

$$u_{\varepsilon, x_0}(x) = \varepsilon^{\frac{2-n}{2}} u\left(\frac{x-x_0}{\varepsilon}\right)$$

$$= \left(\frac{\varepsilon}{\varepsilon^2 + |x-x_0|^2} \right)^{\frac{n-2}{2}},$$

which we will re-encounter later in the context of Yamabe's theorem.

iii) Lions' proof of Thm. 1.5.4 also works in the case of the embedding $\mathring{W}^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ when $\frac{1}{q} = \frac{1}{p} - \frac{k}{n} > 0$ and yields a minimizer of the Sobolev ratio in this case, as well.