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A COUNTEREXAMPLE IN THE VECTORIAL CALCULUS OF VARIATIONS

BERNARD DACOROGNA *and* PAOLO MARCELLINI

Let us consider the integral

$$I(u) = \int_{\Omega} f(\nabla u(x)) \, dx, \quad (1)$$

where Ω is a bounded open set of \mathbb{R}^n , $u: \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, ∇u is the $n \times m$ matrix of the gradient of u , and $f: \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is a continuous function.

When one studies the weak lower semicontinuity of I in a Sobolev space $W^{1,p}$, one is led to consider a necessary condition for f , called *rank one convexity*, and a sufficient condition known as *polyconvexity* (cf. below for the definitions). Both conditions reduce to the ordinary convexity of f if either $m = 1$ or $n = 1$. The two conditions are known not to be equivalent if $m \geq 3$ and $n \geq 3$ (see, Terpstra [4], Serre [3]). Recently Aubert [1] gave an example showing that these conditions are not equivalent if $m = n = 2$; the example is expressed in terms of isotropic functions, i.e.:

$$f(\nabla u) = g(\lambda, \mu), \quad (2)$$

where λ and μ are the eigenvalues of $(\nabla u^T \nabla u)^{\frac{1}{2}}$, the function g being defined (and finite) only if $\lambda, \mu > 0$.

We show here that the example remains valid even if one suppresses the condition of positivity of λ and μ . Moreover, we do not need to consider the representation of f in terms of

g like in (2); therefore we have a direct proof.

First let us define precisely the notions of rank one convexity and polyconvexity; we limit ourselves to the case $m=n=2$.

We denote \mathbb{R}^4 the set of 2×2 matrices ξ . We will use for the matrix ξ also the vectorial notation in terms of components:

$$\xi \equiv (\xi_1, \xi_2, \xi_3, \xi_4). \quad (3)$$

The scalar product and the norm are defined as usual:

$$(\xi, \lambda) = \sum_{i=1}^4 \xi_i \lambda_i, \quad |\xi| = \left(\sum_{i=1}^4 \xi_i^2 \right)^{\frac{1}{2}}. \quad (4)$$

Finally, the determinant of the matrix ξ is

$$\det \xi = \xi_1 \xi_4 - \xi_2 \xi_3. \quad (5)$$

DEFINITION 1: A function $f: \mathbb{R}^4 \rightarrow \mathbb{R}$ is said to be rank one convex if

$$f(t\xi + (1-t)\lambda) \leq tf(\xi) + (1-t)f(\lambda) \quad (6)$$

for every $t \in [0, 1]$ and for every $\xi, \lambda \in \mathbb{R}^4$ with $\text{rank} \{\xi - \lambda\} \leq 1$, i.e. with $\det(\xi - \lambda) = 0$.

If $f \in C^2$ then (6) is equivalent to the well known Legendre-Hadamard (or ellipticity) condition:

$$\sum_{i,j,\alpha,\beta=1}^2 f_{\xi_\alpha^i \xi_\beta^j}(\xi) \mu_\alpha \mu_\beta \eta^i \eta^j \geq 0 \quad (7)$$

for every $\xi = (\xi_\alpha^i) \in \mathbb{R}^4$ (here we use the matricial notation) and for every $\mu, \eta \in \mathbb{R}^2$.

Since a matrix $\lambda \in \mathbb{R}^4$ can be represented in the form $(\mu_1 \eta_1, \mu_1 \eta_2, \mu_2 \eta_1, \mu_2 \eta_2)$ if and only if $\det \lambda = 0$, the Legendre-Hadamard condition (7) is equivalent to:

$$\sum_{i,j=1}^4 f_{\xi_i \xi_j}(\xi) \lambda_i \lambda_j \geq 0 \quad (8)$$

for every $\xi \in \mathbb{R}^4$ and for every $\lambda \in \mathbb{R}^4$ with $\det \lambda = 0$.

Ball in [2] proposed the following:

DEFINITION 2: A function f is said to be rank one convex if there exists a convex function g such that $f(\xi) = g(\xi)$ for every $\xi \in \mathbb{R}^4$.

We note that in

f pol

THEOREM: Let f be a rank one convex function.

Then f is rank one convex.

REMARKS

(i) The function f is not necessarily convex. For example if we restrict f to the set of rank one matrices, we obtain a convex function.

(ii) It is not known if a rank one convex function f as in (11), is weakly p -convex for $p \geq 1$.

PROOF: - Part 1: The function f is rank one convex. To this end, let us consider a rank one matrix ξ . We can write $\xi = \mu \eta^T$ for some $\mu, \eta \in \mathbb{R}^2$. Let us consider the function g , like even in (11), which is convex from below by an affine function.

$$g(\xi_1, \xi_2, \dots)$$

for some $a_i \in \mathbb{R}$ ($i=1, 2, \dots$).

Since $\delta = \det \xi$ and (9) there exists a function h such that

This is absurd. Let us consider a real parameter, we

DEFINITION 2: A function $f: \mathbb{R}^4 \rightarrow \mathbb{R}$ is said to be polyconvex if there exists a convex function $g: \mathbb{R}^5 \rightarrow \mathbb{R}$ so that, for every $\xi \in \mathbb{R}^4$:

$$f(\xi) = g(\xi, \det \xi). \quad (9)$$

We note that in general we have:

$$f \text{ polyconvex} \implies f \text{ rank one convex}. \quad (10)$$

THEOREM: Let f be defined, for $\xi \in \mathbb{R}^4$, by

$$f(\xi) = |\xi|^4 - \frac{4}{\sqrt{3}} |\xi|^2 \det \xi. \quad (11)$$

Then f is rank one convex but not polyconvex.

REMARKS

(i) The function f defined in (11) gives essentially Aubert's example if we restrict to diagonal matrices with positive eigenvalues.

(ii) It is not known if the integral $I(u)$ defined in (1), with f as in (11), is weakly lower semicontinuous in $W^{1,p}$ for some $p > 1$.

PROOF: — Part 1: The fact that f is not polyconvex can be easily seen. To this end, let us assume, for contradiction, that f is polyconvex and that the representation formula in (9) holds. The function g , like every proper convex function, must be bounded from below by an affine function; that is

$$g(\xi_1, \xi_2, \xi_3, \xi_4, \delta) \geq a_0 + \sum_{i=1}^4 a_i \xi_i + a_5 \delta \quad (12)$$

for some $a_i \in \mathbb{R}$ ($i=0, \dots, 5$) and for every $\xi \in \mathbb{R}^4$, $\delta \in \mathbb{R}$.

Since $\delta = \det \xi$ is quadratic with respect to ξ , by (12) and (9) there exists a constant c such that

$$\frac{f(\xi)}{1 + |\xi|^2} \geq c, \quad \forall \xi \in \mathbb{R}^4. \quad (13)$$

This is absurd; in fact, for $\xi \equiv (t, 0, 0, t)$, with t a real parameter, we have $|\xi|^2 = 2t^2$, $\det \xi = t^2$, and thus

$$\frac{f(\xi)}{1+|\xi|^2} = \frac{4t^4 - (8/\sqrt{3})t^4}{1+2t^2} = \frac{4(\sqrt{3}-2)}{\sqrt{3}} \frac{t^4}{1+2t^2}, \quad (14)$$

which goes to $-\infty$ as $t \rightarrow \infty$.

Part 2: To prove that f is rank one convex is more involved and we decompose the proof into three steps. We will show that f satisfies the Legendre-Hadamard condition as in (8).

Step 1: With the aim of computing the quadratic form in (8) we introduce some notation. To every $\xi \in \mathbb{R}^4$ we associate $\hat{\xi} \in \mathbb{R}^4$ defined by:

$$\text{if } \xi \equiv (\xi_1, \xi_2, \xi_3, \xi_4) \quad \text{then } \hat{\xi} \equiv (\xi_4, -\xi_3, -\xi_2, \xi_1). \quad (15)$$

The components of $\hat{\xi}$ will be denoted by $\hat{\xi}_i$, $i = 1, \dots, 4$. We observe that:

$$\begin{cases} |\xi| = |\hat{\xi}| \\ \det \xi = \det \hat{\xi} \\ (\xi, \hat{\xi}) = 2 \det \xi \\ (\xi, \hat{\lambda}) = (\hat{\xi}, \lambda). \end{cases} \quad (16)$$

With this notation we can easily express the gradient of the determinant of ξ :

$$(\det \xi)_{\xi_i} = \hat{\xi}_i \quad \text{for } i = 1, \dots, 4; \quad \text{i.e. } \nabla(\det \xi) = \hat{\xi}. \quad (17)$$

For the second derivatives of the determinant we use the notation

$$\frac{\partial^2}{\partial \xi_i \partial \xi_j} (\det \xi) = \frac{\partial}{\partial \xi_j} (\hat{\xi}_i) = \hat{\delta}_{ij}. \quad (18)$$

Since the quadratic form associated to the matrix of the second derivatives is equal to twice the original quadratic form, we have

$$\sum_{i,j=1}^4 \hat{\delta}_{ij} \lambda_i \lambda_j = \sum_{i,j=1}^4 (\det \xi)_{\xi_i \xi_j} \lambda_i \lambda_j = 2 \det \lambda. \quad (19)$$

Finally

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Finally, as usual, we denote $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$.

Step 2: We compute the quadratic form in (8). Let us begin with the derivatives of f :

$$\begin{aligned} f_{\xi_i} &= \frac{\partial}{\partial \xi_i} \left(|\xi|^4 - \frac{4}{\sqrt{3}} |\xi|^2 \det \xi \right) \\ &= 4|\xi|^2 \xi_i - \frac{8}{\sqrt{3}} \xi_i \det \xi - \frac{4}{\sqrt{3}} |\xi|^2 \hat{\xi}_i. \end{aligned} \quad (20)$$

$$\begin{aligned} f_{\xi_i \xi_j} &= 8 \xi_i \xi_j + 4|\xi|^2 \delta_{ij} - \frac{8}{\sqrt{3}} (\delta_{ij} \det \xi + \xi_i \hat{\xi}_j) \\ &\quad - \frac{4}{\sqrt{3}} (2 \hat{\xi}_i \xi_j + |\xi|^2 \hat{\delta}_{ij}). \end{aligned}$$

Then the quadratic form in (8) is equal to

$$\begin{aligned} \sum_{i,j=1}^4 f_{\xi_i \xi_j} \lambda_i \lambda_j &= 8(\xi, \lambda)^2 + 4|\xi|^2 |\lambda|^2 - \frac{8}{\sqrt{3}} \det \xi |\lambda|^2 \\ &\quad - \frac{16}{\sqrt{3}} (\xi, \lambda) (\hat{\xi}, \lambda) - \frac{8}{\sqrt{3}} |\xi|^2 \det \lambda. \end{aligned} \quad (21)$$

In the case of interest ($\det \lambda = 0$) we have

$$\begin{aligned} \psi(\xi, \lambda) &\stackrel{\text{def}}{=} \frac{1}{4} \sum_{i,j=1}^4 f_{\xi_i \xi_j} \lambda_i \lambda_j = \\ &= |\xi|^2 |\lambda|^2 + 2(\xi, \lambda)^2 - \frac{2}{\sqrt{3}} \det \xi |\lambda|^2 - \frac{4}{\sqrt{3}} (\xi, \lambda) (\hat{\xi}, \lambda). \end{aligned} \quad (22)$$

Step 3: The rank one convexity of $f(\xi)$ is reduced to showing the nonnegativity of $\psi(\xi, \lambda)$ for every ξ, λ , with $\det \lambda = 0$.

Observe first that $\psi(\xi, \lambda)$ is homogeneous of degree 2 in ξ (and in λ). Therefore, in order to show that ψ is nonnegative, it is sufficient to prove that

$$\min_{\xi} \left\{ \psi(\xi, \lambda) : |\xi| = 1 \right\} \geq 0, \quad \forall \lambda : \det \lambda = 0. \quad (23)$$

It is clear that the minimum of $\psi(\cdot, \lambda)$ exists on the manifold $|\xi| = 1$. Let α be a Lagrange multiplier; we will prove that $\psi(\xi_0, \lambda) \geq 0$ for every critical point ξ_0 of the function of ξ :

$$\psi(\xi, \lambda) - \alpha(|\xi|^2 - 1). \quad (24)$$

The gradient (with respect to ξ) of the above function is equal to zero when:

$$|\lambda|^2 \xi_0 + 2(\xi_0, \lambda) \lambda - \frac{1}{\sqrt{3}} |\lambda|^2 \hat{\xi}_0 - \frac{2}{\sqrt{3}} [(\hat{\xi}_0, \lambda) \lambda + (\xi_0, \lambda) \hat{\lambda}] = \alpha \xi_0. \quad (25)$$

Upon multiplication by ξ_0 , bearing in mind that $|\xi_0|^2 = 1$ and that $(\hat{\xi}_0, \xi_0) = 2 \det \xi_0$, we obtain

$$\psi(\xi_0, \lambda) = \alpha. \quad (26)$$

Upon multiplication of (25) first by λ , then by $\hat{\lambda}$, bearing in mind that $(\hat{\lambda}, \lambda) = 2 \det \lambda = 0$ and that $(\hat{\xi}, \hat{\lambda}) = (\xi, \lambda)$, we have

$$\begin{cases} (3|\lambda|^2 - \alpha)(\xi_0, \lambda) - \sqrt{3} |\lambda|^2 (\hat{\xi}_0, \lambda) = 0 \\ -\sqrt{3} |\lambda|^2 (\xi_0, \lambda) + (|\lambda|^2 - \alpha)(\hat{\xi}_0, \lambda) = 0, \end{cases} \quad (27)$$

which leads to either $(\xi_0, \lambda) = (\hat{\xi}_0, \lambda) = 0$, or

$$(3|\lambda|^2 - \alpha)(|\lambda|^2 - \alpha) - 3|\lambda|^4 = \alpha^2 - 4|\lambda|^2 \alpha = 0. \quad (28)$$

In this second case we would have either $\alpha = 0$ or $\alpha = 4|\lambda|^2$, and thus $\psi(\xi_0, \lambda) = \alpha \geq 0$. In the first case $(\xi_0, \lambda) = (\hat{\xi}_0, \lambda) = 0$ we would have

$$\begin{aligned} \psi(\xi_0, \lambda) &= |\lambda|^2 \left(|\xi_0|^2 - \frac{2}{\sqrt{3}} \det \xi_0 \right) \\ &\geq |\lambda|^2 |\xi_0|^2 \left(1 - \frac{1}{\sqrt{3}} \right) \geq 0, \end{aligned} \quad (29)$$

since $2|\det \xi| \leq |\xi|^2$ for every $\xi \in \mathbb{R}^4$. This completes the proof.

REFERENCES

1. G. Aubert, *Thèses et Applications*, Doctorat, Paris (1977).
2. J.M. Ball, *Nonlinear Elasticity* (1977).
3. D. Serre, *J. Math. Pures et Appl.*
4. F.J. Terpstra, *Summen von Matrizen*, M...

REFERENCES

- 1. G. Aubert, *Contributions aux Problèmes du Calcul des Variations et Application à l'Élasticité Nonlinéaire*, Thèse de Doctorat, Paris VI, 1986.
- 2. J.M. Ball, "Convexity conditions and existence theorems in nonlinear elasticity", *Arch. Rat. Mech. Anal.* **67**, pp.337-403, (1977).
- 3. D. Serre, "Formes quadratiques et calcul des variations", *J. Math. Pures et Appl.* **62**, pp.177-196, (1983).
- 4. F.J. Terpstra, "Die Darstellung biquadratischer Formen als Summen von Quadraten mit Anwendung auf die Variationsrechnung", *Math. Ann.* **116**, pp.116-180, (1938).

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