1.2 The Manià example

We present in this section the example due to B. Manià (1934, [37]) of a Lagrangian that exhibits the Lavrentiev phenomenon (1926, [34]). In the next chapter we discuss further this example in relation with the new results obtained ([12], [26]).

We recall that a Lagrangian L exhibits the Lavrentiev phenomenon if the infimum taken over the set of absolutely continuous trajectories $\mathbf{AC}[a,b]$ is strictly lower than the infimum taken over the set of Lipschitzian trajectories $\mathbf{Lip}[a,b]$, with fixed boundary conditions.

Consider the problem of minimize the action

$$\mathcal{I}(x) = \int_0^1 [x^3(t) - t]^2 x'^6(t)dt,$$

on the trajectories x satisfying the boundary conditions $x(0) = 0$, $x(1) = 1$.

Theorem 1.5 (Manià, 1934)**.** *The Lagrangian* $L(t,x,\xi) = (x^3 - t)^2\xi^6$ *exhibits the Lavrentiev phenomenon, i.e.*

$$\inf_{x\in\mathbf{AC}_*[0,1]} \mathcal{I}(x) < \inf_{x\in\mathbf{Lip}_*[0,1]} \mathcal{I}(x),$$

where $\mathbf{AC}_*[0,1] = \{x \in \mathbf{AC}[0,1] : x(0) = 0, x(1) = 1\}$ *and* $\mathbf{Lip}_*[0,1] = \{x \in \mathbf{Lip}[0,1] : x(0) = 0, x(1) = 1\}$.

Proof. By definition, the Lagrangian L and the action $\mathcal{I}$ have non-negative values. By the fact that $\mathcal{I}$ evaluated in $\hat{x}(t) = \sqrt[3]{t}$ is zero, we have that $\hat{x}$ is a minimizer of $\mathcal{I}$ on $\mathbf{AC}_*[0,1]$.

Let x be any trajectory in $\mathbf{Lip}_*[0,1]$ and consider the function $f(t) = \sqrt[3]{t}/2$. By the regularity of x, there exists a real number a in $(0,1)$ such that $x(t) \leq f(t)$, for any t in $[0,a]$, and $x(a) = f(a)$. Hence,

$$[x^3(t) - t]^2\xi^6 \geq [f^3(t) - t]^2\xi^6 = \frac{7^2}{8^2}t^2\xi^6,$$

for any t in $[0,a]$, any ξ in $\mathbb{R}$. By the Hölder inequality, we have

$$\frac{\sqrt[3]{a}}{2} = \int_0^a \frac{\sqrt[3]{t}}{\sqrt[3]{t}}x'(t)dt \leq \left(\int_0^a t^{-2/5}dt\right)^{5/6}\left(\int_0^a t^2x'^6(t)dt\right)^{1/6} = \frac{5^{5/6}}{3^{5/6}}a^{1/2}\left(\int_0^a t^2x'^6(t)dt\right)^{1/6}.$$

We conclude that, for any x in $\mathbf{Lip}_*[0,1]$,

$$\mathcal{I}(x) \geq \int_0^a [x^3(t) - t]^2x'^6(t)dt \geq \frac{7^2 3^5}{8^2 5^5 2^6 a} \geq \frac{7^2 3^5}{8^2 5^5 2^6} > 0.$$

□

Perturbing the Lagrangian of Manià it is possible to construct Lagrangians that exhibit the Lavrantiev phenomenon. This is true for a generic Lagrangian ([5], [16], [34]):

Propositon 1.6. *Let L be a Lagrangian that exhibits the Lavrentiev phenomenon. Suppose that $\mathcal{I} \geq 0$ and that there exists a minimizer $\hat{x}$ with $\mathcal{I}(\hat{x}) = 0$.*

Then, for any action $\mathcal{P}(x) = \int_a^b P(t, x, x')$ such that $\mathcal{P} \geq 0$ and $\mathcal{P}(\hat{x})$ is finite, there exists $\hat{\epsilon} > 0$ such that, for any ϵ in $[0, \hat{\epsilon}]$, the Lagrangian $L + \epsilon P$ exhibits the Lavrentiev phenomenon.

Proof. Let c be a positive constant such that $\mathcal{I}(x) \geq c$, for any Lipschitz trajectory x. Setting $\hat{\epsilon} = c/[2\mathcal{P}(\hat{x})]$, we have, for any ϵ in $[0, \hat{\epsilon}]$, $\mathcal{I}(\hat{x}) + \epsilon\mathcal{P}(\hat{x}) \leq c/2$ and $\mathcal{I}(x) + \epsilon\mathcal{P}(x) \geq c$, for any Lipschitz trajectory x. Hence, $L + \epsilon P$ exhibits the Lavrentiev phenomenon. □

Consider the Lagrangian $P(t, x, \xi) = |\xi|^{5/4}$. From the previous proposition it follows that the problem of the Calculus of Variations with action

$$\int_0^1 \{[x^3(t) - t]^2 x'^6(t) + \epsilon|x'(t)|^{5/4}\} dt,$$

and boundary conditions $x(0) = 0$, $x(1) = 1$, presents the Lavrentiev phenomenon. This is significant because the Lagrangian in this problems is strictly convex in its last variable and with super-linear growth.

The Lagrangian of Manià presents also another interesting phenomenon: the repulsion property ([5], [34]).

Theorem 1.7. *For any sequence of Lipschitz trajectories $\{x_n\}_n$ such that x_n tends to $\hat{x}$, as n tends to ∞, almost everywhere on $[0, 1]$, we have that $\mathcal{I}(x_n)$ diverges to ∞.*

Proof. For any integer n, let a_n in $(0, 1)$ be such that $x_n(t) \leq \sqrt[3]{t}/2$, for any t in $[0, a_n]$, and $x(a_n) = \sqrt[3]{a_n}/2$. By the convergence of $x_n(t)$ to $\hat{x}(t)$, for almost every t in $[0, 1]$, we have that a_n tends to 0.

Using the inequality obtained in the proof of Theorem 1.5, we have

$$\mathcal{I}(x_n) \geq \int_0^{a_n} [x^3(t) - t]^2 x'^6(t) dt \geq \frac{7^2 3^5}{8^2 5^5 2^6 a_n}.$$

Hence, $\mathcal{I}(x_n)$ tends to ∞, as n tends to ∞. □