Quasiconvex functions with subquadratic growth

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We establish the existence of quasiconvex functions which are not convex and which have subquadratic growth at infinity.

We consider variational integrals

$$I(u) = \int_{\Omega} f(\mathrm{D}u(x))\,\mathrm{d}x, \tag{1}$$

defined for (sufficiently regular) functions $u:\Omega \to \boldsymbol{R}^m$. Here Ω is a bounded open subset of $\boldsymbol{R}_n$, $\mathrm{D}u(x)$ denotes the gradient matrix of u at x and f is a continuous function on the space of all real $m \times n$ matrices $M^{m\times n}$. The following conditions on f are related to weak lower semicontinuity of the integral I (see Ball 1978; Morrey 1966).

(i) f is *rank-one convex* if for each matrix $A \in M^{m\times n}$ and each rank-one matrix $B \in M^{m\times n}$ the function $t \to f(A+tB)$ is convex.

(ii) f is *quasiconvex* if for any matrix $A \in M^{m\times n}$ and any smooth function $\varphi:\Omega \to \boldsymbol{R}^m$ compactly supported in Ω the inequality

$$\int_{\Omega} f(A+\mathrm{D}\varphi)\,\mathrm{d}x \geqslant \int_{\Omega} f(A)\,\mathrm{d}x$$

holds true. The class of quasiconvex functions is independent of Ω (see Ball 1978; Morrey 1966).

(iii) f is *polyconvex* if $f(X) =$ convex function of minors of the matrix X. In particular, $f:M^{2\times 2} \to \boldsymbol{R}$ is polyconvex if there exists a convex function $G:M^{2\times 2}\times \boldsymbol{R} \to \boldsymbol{R}$ such that $f(X) = G(X, \det X)$ for each $X \in M^{2\times 2}$.

It is well-known that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). The implication (i) $\Rightarrow$ (ii) is true for quadratic forms, but it is not known whether or not it is true for general f. (See Ball 1978; Morrey 1966.)

There are examples of quadratic forms on $M^{3\times 3}$ showing that (ii) does not imply (iii) (see Ball 1985; Serre 1983; Terpstra 1938). On $M^{2\times 2}$ every quasiconvex quadratic form is polyconvex (Serre 1983; Terpstra 1938). Here we give some examples showing that for general functions on $M^{2\times 2}$ (ii) does not imply (iii).

It is easy to see that every polyconvex function with subquadratic growth at infinity is convex. We aim to show that on $M^{2\times 2}$ there are quasiconvex functions with subquadratic growth at infinity which are not convex.

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We recall that for a continuous function $f: M^{2\times 2} \to \boldsymbol{R}$ its *quasiconvexification* $\mathrm{Q}f$ is defined by

$$\mathrm{Q}f(X) = \inf\left\{\frac{1}{\operatorname{meas}\Omega}\int_{\Omega} f(X+\mathrm{D}\varphi)\,\mathrm{d}x,\ \varphi \text{ is smooth and compactly supported in } \Omega\right\}$$

$\mathrm{Q}f$ does not depend on Ω and if f is bounded below, then $\mathrm{Q}f$ is continuous and quasiconvex (see, for example, Dacorogna 1989).

Theorem. *Let $p > 1$. Let L be a two-dimensional affine subspace of $M^{2\times 2}$ which does not contain any rank-one direction (i.e.* $\operatorname{rank}(A-B) \geqslant 2$ *for any two distinct $A, B \in L$), and let K be a closed subset of L. Let $f: M^{2\times 2} \to \boldsymbol{R}$ be defined by $f(X) = (\operatorname{distance}(X,K))^p$. Then the quasiconvexification $\mathrm{Q}f$ of f satisfies $\mathrm{Q}f(X) > 0$ for any $X \in M^{2\times 2}\backslash K$.*

Proof. Since the convexification $\mathrm{C}f$ of f is clearly positive outside L and $\mathrm{Q}f \geqslant \mathrm{C}f$, it is enough to show that $\mathrm{Q}f(A) > 0$ for any $A \in L\backslash K$. We can assume without loss of generality that $A = 0$. Assuming this, suppose that $\mathrm{Q}f(0) = 0$. Then there is a sequence φ_j of smooth functions $\varphi_j: \Omega \to \boldsymbol{R}^2$ compactly supported in Ω such that

$$\lim_{j\to\infty}\int_{\Omega} f(\mathrm{D}\varphi_j)\,\mathrm{d}x = 0. \tag{2}$$

Let us assume for a moment that

$$L = L_0 = \left\{\begin{pmatrix} s & -t \\ t & s\end{pmatrix},\quad s,t \in \boldsymbol{R}\right\}.$$

In this case

$$\operatorname{distance}(X,K) \geqslant \operatorname{distance}(X,L_0) \geqslant \tfrac{1}{2}(|x_{11}-x_{22}|+|x_{12}+x_{21}|),$$

and using (2) we see that

$$\lim_{j\to\infty}\int_{\Omega}(|\partial_1\varphi_{j1}-\partial_2\varphi_{j2}|^p+|\partial_2\varphi_{j1}+\partial_1\varphi_{j2}|^p)\,\mathrm{d}x = 0. \tag{3}$$

We can write

$$\left.\begin{aligned}\Delta\varphi_{j1} &= \partial_1(\partial_1\varphi_{j1}-\partial_2\varphi_{j2})+\partial_2(\partial_2\varphi_{j1}+\partial_1\varphi_{j2}),\\ \Delta\varphi_{j2} &= \partial_1(\partial_2\varphi_{j1}+\partial_1\varphi_{j2})-\partial_2(\partial_1\varphi_{j1}-\partial_2\varphi_{j2}).\end{aligned}\right\} \tag{4}$$

From (3) we see that the right-hand side of (4) converges to zero strongly in $W^{-1,p}(\Omega)$ and we can use the well-known L^p estimates for the Laplace operator (see, for example, Morrey 1966; Stein 1970) to infer that $\mathrm{D}\varphi_j$ converges to zero strongly in $L^p(\Omega)$. This contradicts (2) and the proof of the theorem in the case $L = L_0$ is finished.

Let us finish the proof of the general case. Recalling that we have assumed $0 \in L$, we notice that we can find two non-singular matrices P, Q such that $P\cdot L\cdot Q^{-1} = L_0$. Indeed, replacing L by $Y^{-1}\cdot L$ where $0 \neq Y \in L$, we can assume $I \in L$. An easy calculation shows that the space $L = \{sI+tA,\ s,t\in\boldsymbol{R}\}$ does not contain any rank one connection if and only if the real Jordan form of A belongs to L_0. The existence of the required P and Q follows.

Let $\tilde{\Omega} = Q(\Omega)$ and let $\tilde{\varphi}_j: \tilde{\Omega} \to \boldsymbol{R}^2$ be defined by $\tilde{\varphi}_j(\tilde{x}) = P\cdot\varphi_j(Q^{-1}\tilde{x})$, where φ_j are the functions from (2). We have $\mathrm{D}\tilde{\varphi}_j(\tilde{x}) = P\cdot\mathrm{D}\varphi_j(Q^{-1}\tilde{x})\cdot Q^{-1}$. Clearly there exists $c > 0$ such that

$$c^{-1}\cdot\operatorname{distance}(X,K) \leqslant \operatorname{distance}(P\cdot X\cdot Q^{-1}, P\cdot K\cdot Q^{-1}) \leqslant c\cdot\operatorname{distance}(X,K)$$

for each $X \in M^{2\times 2}$. Since, moreover, $P \cdot K \cdot Q^{-1} \subset L_0$, we can use (2) to infer that (3) is satisfied with φ_j replaced by $\tilde{\varphi}_j$ and Ω replaced by $\tilde{\Omega}$. We have seen that this implies that $\mathrm{D}\tilde{\varphi}_j$ converges to zero strongly in $L^p(\tilde{\Omega})$. Therefore $\mathrm{D}\varphi_j$ converges to zero strongly in $L^p(\Omega)$ and we again get the contradiction. The proof is finished.

Corollary. *There exist quasiconvex functions on $M^{2\times 2}$ which are not polyconvex.*

Proof. The function f from the preceding theorem is continuous and $\geqslant 0$. Therefore $\mathrm{Q}f$ is continuous, quasiconvex, and $\geqslant 0$ (see, for example, Dacorogna 1989). If $p < 2$ and K is not convex, then $\mathrm{Q}f$ cannot be polyconvex, since it is not convex and has subquadratic growth at infinity.

Remarks. (1) The assertion of the theorem remains true for $p = 1$ if we assume that K is bounded. In this case the proof is more difficult since we cannot use the standard L^p estimates. There are several ways to get over this difficulty. Stefan Müller showed me this simple one. Although the standard L^p estimates fail for $p = 1$ the *weak type* L^1 estimates hold true (see, for example, Stein 1970). Hence under the assumptions of the theorem with $p = 1$ we know that the functions $\mathrm{D}\varphi_j$ converge to zero in the weak L^1 space, i.e. $\lim_{j\to\infty} \alpha \cdot \mathrm{meas}\{x \in \Omega,\ |\mathrm{D}\varphi_j(x)| \geqslant \alpha\} = 0$ for each $\alpha > 0$. In addition to this, from the boundedness of K and (2) we see that

$$\lim_{j\to\infty} \int_{\Omega \cap \{|\mathrm{D}\varphi_j| \geqslant k\}} |\mathrm{D}\varphi_j| \,\mathrm{d}x = 0$$

if we choose k sufficiently large. These two facts imply that $\mathrm{D}\varphi_j$ converge to zero strongly in L^1 and we again get the contradiction.

(2) If K is invariant under transformations $X \to R^+ \cdot X$ and $X \to R \cdot X \cdot R^{-1}$ with $R^+ \in \mathrm{SO}(2)$ and $R \in \mathrm{O}(2)$ then $\mathrm{Q}f$ from the theorem is invariant under the same transformations. An example of such as K is $\mathrm{SO}(2) \cup 2 \cdot \mathrm{SO}(2)$.

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