showed that this property characterizes laminates.

**Theorem 4.16** A compactly supported probability measure \( \nu \in \mathcal{M}(\mathbb{M}^{m \times n}) \) is a laminate if and only if

\[
\langle \nu, f \rangle \geq f(\langle \nu, \text{id} \rangle)
\]

for all rank-1 convex functions \( f : \mathbb{M}^{m \times n} \to \mathbb{R} \). In other words, the laminates supported on a compact set \( K \) are given exactly by \( \mathcal{M}^{\text{rc}}(K) \).

The question raised in the title of this subsection may now be stated more precisely:

Are all gradient Young measures laminates?

In view of Theorem 4.16 this may be concisely stated as

\[
\mathcal{M}^{\text{rc}} = \mathcal{M}^{\text{qc}}.
\]

This would clearly be true if rank-1 convexity implied quasiconvexity. Conversely if \( \mathcal{M}^{\text{rc}} = \mathcal{M}^{\text{qc}} \) then rank-1 convexity would imply quasiconvexity in view of the definition of \( \mathcal{M}^{\text{rc}} \) and the fact that \( f^\#(F) = \inf \{ \langle \nu, f \rangle : \nu \in \mathcal{M}^{\text{qc}}, \langle \nu, \text{id} \rangle = F \} \) (one equality follows from the definition of \( \mathcal{M}^{\text{qc}} \); for the other use Theorem 4.5 (iii) for \( \Omega = (0,1)^n \), extend \( \varphi \) periodically, let \( \varphi_k(x) = k^{-1} \varphi(kx) \) and note that \( \{ D\varphi_k \} \) generates a homogeneous gradient Young measure).

In the next section we discuss Šverák’s example that shows that rank-1 convexity does not imply quasiconvexity if the target dimension satisfies \( m \geq 3 \).

### 4.7 Šverák’s counterexample

**Theorem 4.17** (Šverák [Sv 92a]) Suppose that \( m \geq 3, n \geq 2 \). Then there exists a function \( f : \mathbb{M}^{m \times n} \to \mathbb{R} \) which is rank-1 convex but not quasiconvex.

Using this result Kristensen recently showed that there is no local condition that implies quasiconvexity. This finally resolves, for \( m \geq 3 \), the conjecture carefully expressed by Morrey in his fundamental paper [Mo 52], p. 26: ‘In fact, after a great deal of experimentation, the writer is inclined to think that there is no condition of the type discussed, which involves \( f \) and only a finite number of its derivatives, and which is both necessary and sufficient for quasi-convexity in the general case.’
To state Kristensen’s result let us denote by $\mathcal{F}$ the space of extended real-valued functions $f : M^{m \times n} \to [-\infty, \infty]$. An operator $\mathcal{P} : C^\infty(M^{m \times n}) \to \mathcal{F}$ is called local if the implication

$$f = g \text{ in a neighbourhood of } F \implies \mathcal{P}(f) = \mathcal{P}(g) \text{ in a neighbourhood of } F$$

holds.

**Theorem 4.18** ([Kr 97a]) Suppose that $m \geq 3, n \geq 2$. There exists no local operator $\mathcal{P} : C^\infty(M^{m \times n}) \to \mathcal{F}$ such that

$$\mathcal{P}(f) = 0 \iff f \text{ is quasiconvex.}$$

By contrast, the local operator

$$\mathcal{P}_\ell(f)(F) = \inf\{D^2f(F)(a \otimes b, a \otimes b) : a \in \mathbb{R}^m, b \in \mathbb{R}^n\}$$

characterizes rank-1 convexity. At the end of this subsection we will give an argument of Šverák that proves Theorem 4.18 for $m \geq 6$.

Most research before Šverák’s result focused on choosing a particular rank-1 convex integrand $f$ (e.g., the Dacorogna–Marcellini example given by (4.2)) and trying to prove or disprove that there exists a function $u \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$ and $F \in M^{m \times n}$ such that

$$\int\limits_\Omega f(F + Du)dx < \int\limits_\Omega f(F)dx. \tag{4.22}$$

Šverák’s key idea was to first fix a function $u$ and to look for integrands $f$ that satisfy (4.22) but are rank-1 convex. He made the crucial observation that the linear space spanned by gradients of trigonometric polynomials contains very few rank-1 direction and hence supports many rank-1 convex functions.

To proceed, it is useful to note that quasiconvexity can be defined using periodic test functions rather than functions that vanish on the boundary.

**Proposition 4.19** A continuous function $f : M^{m \times n} \to \mathbb{R}$ is quasiconvex if and only if

$$\int\limits_Q f(F + Du)dx \geq f(F)$$

for all Lipschitz functions $u$ that are periodic on the unit cube $Q$ and all $F \in M^{m \times n}$. 

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Proof. Sufficiency of the condition is clear since it suffices to verify condition (4.1) for $Q$ (see Remark 2 after Definition 4.2). To establish necessity consider a periodic Lipschitz function $u$ and cut-off functions $\varphi_k \in C_0^\infty((-k, k)^n)$ such that $0 \leq \varphi_k \leq 1$, $\varphi_k = 1$ on $(-(k - 1), (k - 1))^n$ and $|D\varphi| \leq C$. If we let $v_k = \varphi_k u, w_k(x) = \frac{1}{k} v_k(kx)$ then quasiconvexity implies that
\[
(k - 1)^n \int_Q f(F + Du)dy \geq \int_{(-k, k)^n} f(F + Dv_k)dx - Ck^{n-1}
\]
\[
= k^n \int_Q f(F + Dw_k)dx - Ck^{n-1} \geq k^n f(F) - Ck^{n-1}.
\]
Division by $k^n$ yields the assertion as $k \to \infty$. \hfill \Box

Proof of Theorem 4.17. Consider the periodic function $u : \mathbb{R}^2 \to \mathbb{R}^3$
\[
u(x) = \frac{1}{2\pi} \begin{pmatrix} \sin 2\pi x^1 \\ \sin 2\pi x^2 \\ \sin 2\pi (x^1 + x^2) \end{pmatrix}.
\]
Then
\[
Du(x) = \begin{pmatrix} \cos 2\pi x^1 & 0 \\ 0 & \cos 2\pi x^2 \\ \cos 2\pi (x^1 + x^2) & \cos 2\pi (x^1 + x^2) \end{pmatrix}
\]
and
\[
L := \text{span}\{Du(x)\}_{x \in \mathbb{R}^2} = \left\{ \begin{pmatrix} r & 0 \\ 0 & s \\ t & t \end{pmatrix} : r, s, t \in \mathbb{R} \right\}.
\]
The only rank-1 lines in $L$ are lines parallel to the coordinate axes. In particular the function $g(F) = -rst$ is rank-1 convex (in fact rank-1 affine) on $L$. On the other hand
\[
\int_{(0,1)^2} g(Du(x)) = -\frac{1}{4} < 0 = g(0). \quad (4.23)
\]
To prove the theorem it only remains to show that $g$ can be extended to a rank-1 convex function on $M^{3 \times 2}$. Whether this is possible is unknown. There is, however, a rank-1 convex function that almost agrees with $g$ in $L$ and this
is enough. Let $P$ denote the orthogonal projection onto $L$ and consider the quartic polynomial

$$f_{e,k}(F) = g(PF) + \epsilon(|F|^2 + |F|^4) + k|F - PF|^2.$$

We claim that for every $\epsilon > 0$ there exists a $k(\epsilon) > 0$ such that $f_{e,k(\epsilon)}$ is rank-1 convex. Suppose otherwise. Then there exists an $\epsilon > 0$ such that $f_{e,k}$ is not rank-1 convex for any $k > 0$. Hence there exist $F_k \in M^{m \times n}$, $a_k \in \mathbb{R}^m$, $b_k \in \mathbb{R}^n$, $|a_k| = |b_k| = 1$ such that

$$D^2 f_{e,k}(F_k)(a_k \otimes b_k, a_k \otimes b_k) \leq 0.$$

Now

$$D^2 f_{e,k}(F)(X, X) =$$

$$D^2 g(PF)(PX, PX) + 2\epsilon|X|^2 + \epsilon(|F|^2|X|^2 + 8|F : X|^2) + k|X - PX|^2.$$

The term $D^2 g(PF)$ is linear in $F$ while the third term on the right hand side is quadratic and positive definite. Hence $F_k$ is bounded as $k \to \infty$, and passing to a subsequence if needed, we may assume $F_k \to \tilde{F}, a_k \to \tilde{a}, b_k \to \tilde{b}$. Since $D^2 f_{e,k} \geq D^2 f_{e,j}$ for $k \geq j$ we deduce

$$D^2 g(P\tilde{F})(P\tilde{a} \otimes \tilde{b}, P\tilde{a} \otimes \tilde{b}) + 2\epsilon + \epsilon(|\tilde{a} \otimes \tilde{b} - P\tilde{a} \otimes \tilde{b}|^2 \leq 0 \forall j.$$ (4.24)

Thus $P(\tilde{a} \otimes \tilde{b}) = \tilde{a} \otimes \tilde{b}$, i.e., $\tilde{a} \otimes \tilde{b} \in L$. Therefore $t \mapsto g(P(\tilde{F} + t\tilde{a} \otimes \tilde{b}))$ is affine, and the first term in (4.24) vanishes. This yields the contradiction $\epsilon \leq 0$.

Thus there exist $k(\epsilon)$ such that $f_\epsilon := f_{e,k(\epsilon)}$ is rank-1 convex. By (4.23) and the definition of $u$, the function $f_\epsilon$ is not quasiconvex as long as $\epsilon > 0$ is sufficiently small.

An immediate consequence of Šverák’s result and the considerations in the previous section is that there exist gradient Young measures that are not laminates. In fact the measure $\nu$ defined by averaging $\delta_{Du(x)}$, i.e.

$$\langle \nu, h \rangle = \int_{(0,1)^2} h(Du(x))dx,$$  

$$\forall h \in C_0(M^{m \times n}),$$

provides an example, since $\langle \nu, f_\epsilon \rangle = \langle \nu, g \rangle + C\epsilon < f_\epsilon(\langle \nu, id \rangle)$ (for small $\epsilon > 0$).
The following modification, due to James, provides an even simpler example and a nice illustration of the failure of quasiconvexity for $g$ (or more