5. The Dirichlet Problem for the Equation of Constant Mean Curvature

Another borderline case of a variational problem is the following: Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with generic point $z = (x, y)$ and let $u_0 \in C^0(\overline{\Omega}; \mathbb{R}^3)$, $H \in \mathbb{R}$ be given. Find a solution $u \in C^2(\Omega; \mathbb{R}^3) \cap C^0(\overline{\Omega}; \mathbb{R}^3)$ to the problem

\begin{align}
\Delta u &= 2H u_x \wedge u_y \quad \text{in } \Omega, \\
u &= u_0 \quad \text{on } \partial \Omega.
\end{align}

(5.1)
(5.2)

Here, for $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in \mathbb{R}^3$, $a \wedge b$ denotes the wedge product $a \wedge b = (a_2b_3 - b_2a_3, a_3b_1 - b_3a_1, a_1b_2 - b_1a_2)$ and, for instance, $u_x = \frac{\partial}{\partial x} u$. (5.1) is the equation satisfied by surfaces of mean curvature $H$ in conformal representation.

Surprisingly, (5.1) is of variational type. In fact, solutions of (5.1) may arise as “soap bubbles”, that is, surfaces of least area enclosing a given volume. Also for prescribed Dirichlet data, where a geometric interpretation of (5.1) is impossible, we may recognize (5.1) as the Euler-Lagrange equations associated with the variational integral

$$E_H(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dz + \frac{2H}{3} \int_\Omega u \cdot u_x \wedge u_y \, dz.$$

For smooth “surfaces” $u$, the term

$$V(u) := \frac{1}{3} \int_\Omega u \cdot u_x \wedge u_y \, dz$$

may be interpreted as the algebraic volume enclosed between the “surface” parametrized by $u$ and a fixed reference surface spanning the “curve” defined by the Dirichlet data $u_0$; see Figure 5.1.

Indeed, computing the variation of the volume $V$ at a point $u \in C^2(\Omega; \mathbb{R}^3)$ in direction of a vector $\varphi \in C_0^\infty(\Omega; \mathbb{R}^3)$, we obtain

$$3 \frac{d}{d\varepsilon} V(u + \varepsilon \varphi)\big|_{\varepsilon=0} = \int_\Omega (\varphi \cdot u_x \wedge u_y + u \cdot \varphi_x \wedge u_y + u \cdot u_x \wedge \varphi_y) \, dz$$

$$= 3 \int_\Omega \varphi \cdot u_x \wedge u_y \, dz + \int_\Omega \varphi \cdot (u \wedge u_{xy} + u_{xy} \wedge u) \, dz,$$

and the second integral vanishes by anti-symmetry of the wedge product. Hence critical points $u \in C^2(\Omega; \mathbb{R}^3)$ of $E$ solve (5.1).
Small Solutions

Since $V$ is cubic, in $E_H$ the Dirichlet integral dominates if $u$ is “small” and we can expect that for “small data” and “small” $H$ a solution of (5.1), (5.2) can be obtained by minimizing $E_H$ in a suitable convex set. Generalizing earlier results by Heinz [1] and Werner [1], Hildebrandt [2] has obtained the following result, which is conjectured to give the best possible bounds for the type of constraint considered:

5.1 Theorem. Suppose $u_0 \in H^{1,2} \cap L^\infty(\Omega; \mathbb{R}^3)$. Then for any $H \in \mathbb{R}$ such that

$$\|u_0\|_{L^\infty} \cdot |H| < 1,$$

there exists a solution $u \in u_0 + H^{1,2}_0(\Omega; \mathbb{R}^3)$ of (5.1), (5.2) such that

$$\|u\|_{L^\infty} \leq \|u_0\|_{L^\infty}.$$

The solution $u$ is characterized by the condition

$$E_H(u) = \min \{E_H(v) ; v \in u_0 + H^{1,2}_0 \cap L^\infty(\Omega, \mathbb{R}^3), \|v\|_{L^\infty}\|H\| \leq 1 \}.$$

In particular, $u$ is a relative minimizer of $E_H$ in $u_0 + H^{1,2}_0 \cap L^\infty(\Omega; \mathbb{R}^3)$.

5.2 Remark. Working with a different geometric constraint Wente [1; Theorem 6.1] and Steffen [1; Theorem 2.2] prove the existence of a relative minimizer provided

$$E_0(u_0)H^2 < \frac{2}{3} \pi,$$

where

$$E_0(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dz$$