24.04.2013
Calculus of Variations
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Exercise sheet 1

Exercise 1 (Higher dimensional Catenoid)

Given a function $u : [a,b] \to \mathbb{R}_+$ and $n \in \mathbb{N}$, $n \geq 2$ the functional

$$\mathcal{A}_n(u) := \omega_{n-1} \int_a^b u^{n-1}\sqrt{1+|u'|^2}dx$$

(ω_{n-1} = surface area of $\mathbb{S}^{n-1} \subset \mathbb{R}^n$) is the surface area of the n-dimensional surface of revolution $\Gamma : [a,b] \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}$, $\Gamma(x,\omega) := (x, u(x)\omega)$.

1. Calculate the strong form of the Euler-Lagrange equation for $\mathcal{A}_n$.
2. Show that under the additional assumption $u' > 0$ we get

$$cG\left(\frac{u(x)}{c}\right) = x - x_0$$

for suitable $c > 0, x_0 \in \mathbb{R}$ where

$$G(y) := \int_1^y \frac{1}{\sqrt{u^{2(n-1)} - 1}}du$$

for $y \geq 1$. (Hint: Noether's theorem!)

3. Show that G is strictly monotone and bounded if $n > 2$ but unbounded if $n = 2$. Sketch u!

Exercise 2 (First variation)

For a bounded domain $\Omega \subset \mathbb{R}^n$ let $F(x,y) : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ be measurable in x for fixed y, C^1 in the variables y for fixed x such that there is a constant $C < \infty$ with

$$|F(x,y)| \leq C(1+|y|^q), \quad |F_{y^k}(x,y)| \leq C(1+|y|^{q-1})$$

for a.e. $(x,y) \in \Omega \times \mathbb{R}^m$. For $u \in L^q(\Omega)$ we set

$$\mathcal{F}(v) := \int_\Omega F(x, v(x))dx$$

1. Show that the first variation of $\mathcal{F}$ at $u \in L^q(\Omega)$ in direction $h \in L^q(\Omega)$

$$\delta\mathcal{F}(u;h) := \frac{d}{d\tau}\mathcal{F}(u+\tau h)\bigg|_{\tau=0}$$

exists and calculate its value.

* 2. Show that the function $u \to \delta\mathcal{F}(u;\cdot)$ is a continuous map from $L^q(\Omega)$ to the space $(L^q(\Omega))^*$.

Exercise 3 (Natural boundary conditions)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary (so that we have Gauss's divergence theorem at our disposal) $F(x, y, p) \in C^2(\Omega \times \mathbb{R}^m \times \mathbb{R}^{n \times m})$ and

$$\mathcal{F}(v) := \int_\Omega F(x, v(x), Dv(x))dx.$$

Assume that $u \in C^2(\overline{\Omega})$ is a local minimizer of $\mathcal{F}$ in $C^2(\overline{\Omega})$, i.e. assume that there is a $\delta > 0$ such that

$$\mathcal{F}(v) \geq \mathcal{F}(u), \forall v \in C^2(\Omega) \text{ with } \|u - v\|_{C^2} \leq \delta.$$

1. Deduce that

$$\int_{\partial\Omega} \nu_j F_{p_j^k}(x, u, Du) h^k(x) dS + \int_\Omega \left(-\partial_j F_{p_j^k}(x, u, Du) + F_{p_j^k}(x, u, Du) \right) h^k dx = 0$$

for all $h \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ where ν is the outward pointing unit normal along $\partial\Omega$.

2. Show that

$$\nu_i F_{p_i^k}(x, u, Du) = 0 \text{ on } \partial\Omega$$

This is called *natural boundary condition.*

3. Let $m = 1$. Derive the natural boundary conditions and the Euler-Lagrange equation in the case of $F(x, y, p) = \frac{1}{2}|p|^2 + yf(x)$ and $F(x, y, p) = \sqrt{1 + |p|^2} + yf(x)$ where $f \in C^1(\overline{\Omega})$.