Exercise 1 (Higher dimensional Catenoid)

Given a function \( u : [a,b] \to \mathbb{R}_+ \) and \( n \in \mathbb{N}, n \geq 2 \) the functional

\[
A_n(u) := \omega_{n-1} \int_a^b u^{n-1} \sqrt{1 + |u'|^2} \, dx
\]

(\( \omega_{n-1} \) = surface area of \( S^{n-1} \subset \mathbb{R}^n \)) is the surface area of the \( n \)-dimensional surface of revolution \( \Gamma : [a,b] \times S^{n-1} \to \mathbb{R}^{n+1}, \Gamma(x,\omega) := (x,u(x)\omega) \).

1. Calculate the strong form of the Euler-Lagrange equation for \( A_n \).
2. Show that under the additional assumption \( u' > 0 \) we get

\[
\frac{cG \left( \frac{u(x)}{c} \right)}{c} = x - x_0
\]

for suitable \( c > 0, x_0 \in \mathbb{R} \) where

\[
G(y) := \int_1^y \frac{1}{\sqrt{u^{2(n-1)} - 1}} \, du
\]

for \( y \geq 1 \). (Hint: Noether’s theorem)
3. Show that \( G \) is strictly monotone and bounded if \( n > 2 \) but unbounded if \( n = 2 \).

Sketch \( u \)!

Exercise 2 (First variation)

For a bounded domain \( \Omega \subset \mathbb{R}^m \) let \( F(x,y) : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathbb{R} \) be measurable in \( x \) for fixed \( y \), \( C^1 \) in the variables \( y \) for fixed \( x \) such that there is a constant \( C < \infty \) with

\[
|F(x,y)| \leq C(1 + |y|^q), \quad |F_y(x,y)| \leq C(1 + |y|^{q-1})
\]

for a.e. \( (x,y) \in \Omega \times \mathbb{R}^m \). For \( u \in L^q(\Omega) \) we set

\[
\mathcal{F}(v) := \int_\Omega F(x,v(x)) \, dx
\]

1. Show that the first variation of \( \mathcal{F} \) at \( u \in L^q(\Omega) \) in direction \( h \in L^q(\Omega) \)

\[
\delta \mathcal{F}(u;h) := \left. \frac{d}{d\tau} \mathcal{F}(u + \tau h) \right|_{\tau=0}
\]

exists and calculate its value.

* 2. Show that the function \( u \to \delta \mathcal{F}(u;\cdot) \) is a continuous map from \( L^q(\Omega) \) to the space \( (L^q(\Omega))^* \).
Exercise 3  (Natural boundary conditions)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^1$ boundary (so that we have Gauss’s divergence theorem at our disposal) $F(x, y, p) \in C^2(\Omega \times \mathbb{R}^m \times \mathbb{R}^{n \times m})$ and

$$\mathcal{F}(v) := \int_{\Omega} F(x, v(x), Dv(x)) dx.$$  

Assume that $u \in C^2(\Omega)$ is a local minimizer of $\mathcal{F}$ in $C^2(\Omega)$, i.e. assume that there is a $\delta > 0$ such that

$$\mathcal{F}(v) \geq \mathcal{F}(u), \forall v \in C^2(\Omega) \text{ with } \|u - v\|_{C^2} \leq \delta.$$

1. Deduce that

$$\int_{\partial \Omega} \nu_j F_{p^k_j}(x,u,Du) h^k(x) dS + \int_{\Omega} \left(-\partial_j F_{p^k_j}(x,u,Du) + F_{p^k_j}(x,u,Du)\right) h^k dx = 0$$

for all $h \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ where $\nu$ is the outward pointing unit normal along $\partial \Omega$.

2. Show that

$$\nu_i F_{p^k_i}(x,u,Du) = 0 \text{ on } \partial \Omega$$

This is called natural boundary condition.

3. Let $m = 1$. Derive the natural boundary conditions and the Euler-Lagrange equation in the case of $F(x, y, p) = \frac{1}{2} |p|^2 + yf(x)$ and $F(x, y, p) = \sqrt{1 + |p|^2} + yf(x)$ where $f \in C^1(\Omega)$. 
