11.05.2013
Calculus of Variations
Dr. Simon Blatt

Exercise sheet 2

Exercise 1 (Some useful inequalities)

1. Show that for $\alpha \in (0,1]$ and $a, b \in \mathbb{R}^+$ we have
$$2^{\alpha-1}(a^\alpha + b^\alpha) \leq (a+b)^\alpha \leq a^\alpha + b^\alpha.$$
(Hint: Use the concavity of $x \to x^\alpha$ on $\mathbb{R}^+$ for the first inequality. For the second discuss the function $g(x) := a^\alpha + x^\alpha - (a+x)^\alpha$ on $[0,\infty)$.)
2. For $p \in [1,\infty)$ and $a, b \in \mathbb{R}^+$ we have
$$(a^p + b^p) \leq (a+b)^p \leq 2^{p-1}(a^p + b^p).$$

Exercise 2 (The theorem of Egorov)

Let $u_m : \Omega \to \mathbb{R}$, $\Omega \subset\subset \mathbb{R}^n$, be a sequence of measurable function converging pointwise to $u : \Omega \to \mathbb{R}$.

1. Show that for every $\varepsilon > 0$ there is a is a set Ω_ε such that $\mathcal{L}^n(\Omega - \Omega_\varepsilon) \leq \varepsilon$ such that u_m converges uniformly to u on Ω_ε.
2. Show that u_m converges to u in measure, i.e. that
$$\mathcal{L}^n(\{|u_m - u| > \varepsilon\}) \to 0$$
for all $\varepsilon > 0$.

Exercise 3 (Weak solutions)

Let $\Omega \subset \mathbb{R}^n$ be a domain, $a_{ij} \in W^{1,\infty}_{loc}(\Omega)$, $b_i, c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$.

1. Show that $u \in W^{2,2}_{loc}(\Omega)$ is a weak solution of
$$-\partial_i(a_{ij}\partial_j u) + b_i \partial_i u + cu = f$$
i.e. that
$$\int_\Omega (a_{ij}\partial_i u \partial_j \varphi + b_i \partial_i u \varphi + cu\varphi)\, dx = \int_\Omega f\varphi dx \qquad \forall \varphi \in C^\infty_c(\Omega)$$
if and only if u satisfies the equation
$$-a_{ij}\partial_{ij}u - \partial_i a_{ij}\partial_j u + b_i \partial_i u + cu = f$$
pointwise for almost every $x \in \Omega$.

Exercise 4 (Vitali's theorem)

Show that the following are equivalent (or: find a proof in the literature ;-)):

1. $f_m \to f$ in $L^1(\mathbb{R}^n)$.
2. The functions f_m are uniformly integrable und $f_n \to f$ in measure.