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Calculus of Variations
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Exercise sheet 3

Exercise 1 (Dirichlet's energy and Plateau's problem)

Proof that for $\Omega \subset\subset \mathbb{R}^2$ and $u \in W^{1,2}(\Omega, \mathbb{R}^3)$

$$\mathcal{D}(u) = \frac{1}{2}\int_\Omega |Du|^2 dz \geq \int_\Omega \sqrt{|\partial_x u|^2 |\partial_y u|^2 - (\partial_x u \cdot \partial_y u)^2} = \mathcal{A}(u)$$

with equality if and only if u is conformal, i.e. if $|\partial_x u| = |\partial_y u|$ and $\partial_x u \cdot \partial_y u = 0$.

Exercise 2 (Weak convergence of minors)

Try to fill in the missing details in the proof of Lemma 4.3, i.e. show the induction step $m \to (m+1)$ for all $m = 1, \dots n-2$, i.e. show that if all the minors up of order m converge weakly in $L^{p/m}$ then th minors of order m converge weakly.(Hint: As in the lecture, derive a divergence structure for the minors.)

Exercise 3 (Concentration effects)

Prove the following statements:

1. Let (u_k) be a bounded sequence in $L^p(\mathbb{R}^n)$

$$\int_{B_R(0)} |u_k|^p dx \to 0 \quad \forall R > 0.$$

   Then $u_k \rightharpoonup 0$ weakly in $L^p(\mathbb{R}^n)$ as $k \to \infty$.

2. Let $u \in W^{1,2,}(\mathbb{R}^n)$ and $u_r(x) := \frac{1}{r^{(n-2)/2}} u(\frac{x}{r})$. Show that then

$$|\nabla u_r|^2 \rightharpoonup \|\nabla u\|_{L^2} \delta_0 \qquad |u_r|^{2^*} \rightharpoonup \|u\|_{L^{2^*}} \delta_0$$

   in the sense of distributions as $r \to 0$ where δ_0 is the Dirac delta function.

Exercise 4 (Concentration Compactness)

Try to reprove Theorem 5.1 applying the concentration compactness lemma Theorem 5.4 to $\mu_k := |u_k|^p dx$ for a minimizing sequence u_k!