Exercise sheet 3

Exercise 1 (Dirichlet’s energy and Plateau’s problem)
Proof that for \( \Omega \subset \subset \mathbb{R}^2 \) and \( u \in W^{1,2}(\Omega, \mathbb{R}^3) \)

\[
D(u) = \frac{1}{2} \int_\Omega |Du|^2 \, dz \geq \int_\Omega \sqrt{|\partial_x u|^2 |\partial_y u|^2 - (\partial_x u \cdot \partial_y u)^2} = A(u)
\]

with equality if and only if \( u \) is conformal, i.e. if \( |\partial_x u| = |\partial_y u| \) and \( \partial_x u \cdot \partial_y u = 0 \).

Exercise 2 (Weak convergence of minors)
Try to fill in the missing details in the proof of Lemma 4.3, i.e. show the induction step \( m \to (m+1) \) for all \( m = 1, \ldots, n-2 \), i.e. show that if all the minors up of order \( m \) converge weakly in \( L^{p/m} \) then the minors of order \( m \) converge weakly. (Hint: As in the lecture, derive a divergence structure for the minors.)

Exercise 3 (Concentration effects)
Prove the following statements:
1. Let \( (u_k) \) be a bounded sequence in \( L^p(\mathbb{R}^n) \)

\[
\int_{B_R(0)} |u_k|^p \, dx \to 0 \quad \forall R > 0.
\]

Then \( u_k \to 0 \) weakly in \( L^p(\mathbb{R}^n) \) as \( k \to \infty \).
2. Let \( u \in W^{1,2}(\mathbb{R}^n) \) and \( u_r(x) := \frac{1}{r^{(n-2)/2}} u\left( \frac{x}{r} \right) \). Show that then

\[
|\nabla u_r|^2 \to \|\nabla u\|_{L^2} \delta_0 \quad |u_r|^2 \to \|u\|_{L^{2^*}} \delta_0
\]

in the sense of distributions as \( r \to 0 \) where \( \delta_0 \) is the Dirac delta function.

Exercise 4 (Concentration Compactness)
Try to reprove Theorem 5.1 applying the concentration compactness lemma Theorem 5.4 to \( \mu_k := |u_k|^p \, dx \) for a minimizing sequence \( u_k \!)