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Calculus of Variations
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Exercise sheet 5

Exercise 1 (Palais-Smale condition)

1. Assume that $\mathcal{F} \in C^1(\mathbb{R}^n)$ is such that $|\mathcal{F}| + |\mathcal{F}'|$ is coercive. Prove that then $\mathcal{F}$ satisfies the Palais-Smale condition at all energy levels $\beta \in \mathbb{R}$.
2. Suppose that $\mathcal{F} \in C^1(X)$ as the property that every Palais-Smale sequence is bounded and
$$\mathcal{F}'(u) = L + K(u)$$
where $L : X \to X'$ is a fixed isomorphism and $K : X \to X'$ is compact. Then $\mathcal{F}$ satisfies the Palais-Smale condition.

Exercise 2 (Mountain Pass Lemma)

Let us assume that $\mathcal{F} \in C^1(X)$ satisfies the Palais-Smale condition at all energy levels and the there are two distinct local minimizers $u_0, u_1 \in X$ of $\mathcal{F}$. Show that there is at least a third critical point.

Exercise 3 (Palais-Smale condition / the critical case)

In this exercise we want to show that
$$\mathcal{F}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx$$
for $p = 2^* = \frac{2n}{n-2}$ does not satisfy the Palais Smale condition at a certain energy level. For that we look at the functions
$$w(x) := \frac{1}{(1+|x^2|)^{\frac{n-2}{2}}}$$
(These are the function that *realize the best Sobolev constant* on $\mathbb{R}^n$, i.e. $S\|w\|^2_{L^{2^*}} = \|\nabla w\|^2_{L^2}$.)

1. Show that
$$\Delta w + |w|^{2^*-2} w = 0$$
for all $x \in \mathbb{R}^n$.
2. Let us assume that $0 \in \Omega \subset\subset \mathbb{R}^n$ is an open domain. Show that the sequence of functions
$$\tilde{w}_k(x) := k^{\frac{n-2}{2}} \varphi(x) w\left(\frac{x - x_0}{k}\right)$$
for k large enough is a Palais-Smale sequence such that
$$|\nabla w_k|^2 dx \rightharpoonup \int_{\mathbb{R}^n} |\nabla w|^2 dx \delta_{\{x=x_0\}} \text{ weak-}*$$
$$|w_k|^{2^*} dx \rightharpoonup \int_{\mathbb{R}^n} |w|^{2^*} dx \delta_{\{x=x_0\}} \text{ weak-}*$$
in the space of finite measures.