Exercise sheet 5

Exercise 1  (Palais-Smale condition)

1. Assume that \( F \in C^1(\mathbb{R}^n) \) is such that \(|F| + |F'|\) is coercive. Prove that then \( F \) satisfies the Palais-Smale condition at all energy levels \( \beta \in \mathbb{R} \).

2. Suppose that \( F \in C^1(X) \) as the property that every Palais-Smale sequence is bounded and

\[
F'(u) = L + K(u)
\]

where \( L : X \to X' \) is a fixed isomorphism and \( K : X \to X' \) is compact. Then \( F \) satisfies the Palais-Smale condition.

Exercise 2  (Mountain Pass Lemma)

Let us assume that \( F \in C^1(X) \) satisfies the Palais-Smale condition at all energy levels and the there are two distinct local minimizers \( u_0, u_1 \in X \) of \( F \). Show that there is at least a third critical point.

Exercise 3  (Palais-Smale condition / the critical case)

In this exercise we want to show that

\[
F(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p} \int_{\Omega} |u|^p dx
\]

for \( p = 2^* = \frac{2n}{n-2} \) does not satisfy the Palais Smale condition at a certain energy level. For that we look at the functions

\[
w(x) := \frac{1}{(1 + |x|^2)^{\frac{n-2}{2}}}
\]

(These are the function that realize the best Sobolev constant on \( \mathbb{R}^n \), i.e. \( S \|w\|_{L^{2^*}}^2 = \|\nabla w\|_{L^2}^2 \).)

1. Show that

\[
\Delta w + |w|^{2^*-2}w = 0
\]

for all \( x \in \mathbb{R}^n \).

2. Let us assume that \( 0 \in \Omega \subset \subset \mathbb{R}^n \) is an open domain. Show that the sequence of functions

\[
\tilde{w}_k(x) := k^{\frac{n-2}{2}} \varphi(x) w \left( \frac{x - x_0}{k} \right)
\]

for \( k \) large enough is a Palais-Smale sequence such that

\[
|\nabla w_k|^2 dx \to \int_{\mathbb{R}^n} |\nabla w|^2 dx \delta_{\{x = x_0\}} \text{ weak-*}
\]

\[
|w_k|^{2^*} dx \to \int_{\mathbb{R}^n} |w|^{2^*} dx \delta_{\{x = x_0\}} \text{ weak-*}
\]

in the space of finite measures.