Calculus of Variations

Tuesday, 8 - 9:30 am  10G 201
Wednesday, 9:45 - 11:15 am  Vener Hörsaal

No exercise class but (in regularly) exercise sheets.

Topic: Study of extremals and critical points of a functional

\[ F : X \rightarrow \mathbb{R} \]

where \( X \) is something of infinite dimension like a function space, space of curves or other geometric objects and \( F \) for example the length of the curve, area of a surface, etc.

One of the oldest problems and still in the focus of today's research is

Dido's Problem: Among all curves of fixed length find the curve which is surrounding a domain of maximal area.
Vergil, Aeneis: Dido was promised as much land as she can surround by the skin of a bull

\[ \text{founded Carthage.} \]

Talestai: The farmer got for 100 Rubel "en Tage Land" - as much as he can surround within a day.

(Unfortunately, he did not go the right way and so exhausted after the day that he died.)

Solution suggested by Jakob Steiner:

He shows that the optimal shape must be a circle in the following way.

**Step 1:** The optimal curve C must be convex. Otherwise we would find points P, Q on C as in the picture.

Reflecting the part C1 of the curve between P and Q we would increase the surrounded area.
Step 2: We select two points $P$ and $Q$ on $C$ which divide the curve into two parts $C^1$ and $C^2$ of equal length.

Then the area $A_1$, surrounded by $C_1$ and $PQ$ must be the same as the area $A_2$ surrounded by $C_2$ and $PQ$.

Proof: Assume consisting of $A_2 > A_1$. Then the curve $+C_2$ and the projection of $C_2$ along $PQ$ surrounds more area than $C_1$.

Step 3: Let $P$ & $Q$ be as in step 2 and $A = C_1$. Then the angle between $PA$ and $AQ$ must be $90^\circ$.

Proof: We construct curve surrounding a larger area, if $\alpha \neq 90^\circ$. 
Since
\[ \theta = \frac{\pi}{2} \text{ or } \theta = \frac{3\pi}{2} \]
we get a curve surrounding more area.

Theorems
\[ \Rightarrow C_1 \text{ and } C_2 \text{ are semi-circles}. \]

Has Starem given a complete proof of the fact that the circle maximizes area? No, since he assumes without proof that the maximum is attained.

We will concentrate on the question of existence of minimizers and critical points (whatever that is!)}
* Mariano Giaquinta, Stefan Hildebrandt
  "Calculus of Variations I", Springer, 2004 (mainly what is called "the
  indirect method of Calc. Var. ")

* Michael Struwe, "Variational Methods",
  Springer, 4 Series of Modern
  Surveys in Mathematics, 2007
  (both are available
  as ebook for KIT
  students)

* Bernard Dacorogna, "Direct Methods
  in the Calculus of Variations",
  Springer, Applied Mathematical Sciences, 78, 1985

* Antonio Ambrosetti and Andrea
  Malchiodi, "Nonlinear analysis and
  semilinear elliptic problems",
  Cambridge Studies in advanced
  mathematics
A rough and tentative outline of this class

§1 Examples and necessary conditions

§2 Functional analytic background

§2.1 Differentiation on Banach spaces

§2.2 Function spaces

(\text{Hölder, Sobolev } \& \text{(BV)})

§2.3 Regularity theory for linear PDE

§3 The direct method

§3.1 Examples

§3.2 Constraints

§3.3 Compensated compactness

§3.4 Concentrated compactness

§4 Min-max constructions and Saddle points
§ 1 Examples and (some) necessary conditions

The problem we will look at will be of the following type:

Give a domain $\Omega \subset \mathbb{R}^m$
(i.e. $\Omega$ connected, open & bounded)
boundary values $\phi \in C^1(\overline{\Omega}, \mathbb{R})$
and $F(x, y, p; \Omega) \in C^1(\partial \Omega \times \mathbb{R}^m \times \mathbb{R}^m)$

find a minimizer of $F(u) = \int_\Omega F(x, u, Du) \, dx$
on the set

$M := \{ v \in C^1(\overline{\Omega}, \mathbb{R}^m) : u(x) = \phi(x) \quad \forall x \in \partial \Omega \}$

Example: (i) $m = 1$, $F(x, y, p; \Omega) = \sqrt{1 + |p|^2}$

$\Rightarrow \quad F(u) = \int_\Omega \sqrt{1 + \nabla u^2} \, dx = \int_\Omega \sqrt{1 + \nabla^2 u^2} \, dx$

area of the graph of $u$. 
\( (ii) \quad F(x, y, p) = 1p_1^2 \)

\[ \Rightarrow \mathcal{E}(u) = \int 1 \, Du_1^2 \, dx \]

= Dirichlet energy

(or \( F(x, y, p) = 1p_1^q \))

A first necessary condition for \( u \) being such a minimiser is the vanishing of the first variation (substitute of the first derivative):

**Definition 1.1:** Let \( u \in C^1(\overline{\Omega}, \mathbb{R}^m) \) and \( h \in C^0(\overline{\Omega}, \mathbb{R}^m) \). If the limit

\[ \mathcal{E}'(u; h) = \lim_{\varepsilon \to 0} \frac{\mathcal{E}(u + \varepsilon h) - \mathcal{E}(u)}{\varepsilon} \]

exists it is called the first variation of \( \mathcal{E} \) at the point \( u \) in direction \( h \).

**Theorem 1.2 (Euler–Lagrange equation)**

In our situation the first variation of \( \mathcal{E} \) exists for all \( h \in C_c^1(\Omega, \mathbb{R}^m) \) and

\[ \mathcal{E}'(u; h) = \int \left( \sum_{i=1}^{n} \left( F_{u_i}^i(x, u, Du) \frac{\partial}{\partial u_i} h_i + \sum_{i=1}^{n} F_{u_i}^i (x, u(x), Du(x)) h_i \right) dx \right) \]
If \( u \) is a minimizer, then
\[
\delta \mathcal{F}(u; h) = 0 \quad \forall h \in C_c^1(\Omega, \mathbb{R}^n)
\]
especially
\[
\int \left( \sum_{i=1}^n F_{p_i^x}(x, u, Du) \, d\mathbf{h} + \sum_{i=1}^n \sum_{j=1}^m F_{p_i^y}(x, u, Du) \, d\mathbf{h} \right) dx
\]
\[
= 0 \quad \forall k = 1, \ldots, m
\]
\( h \in C_c^1(\Omega, \mathbb{R}) \).

(weak form of the Euler-Lagrange equation)

If furthermore \( F \in C^2(\Omega \times \mathbb{R}^n \times \mathbb{R}^{m \times m}) \) and \( u \in C^2(\Omega, \mathbb{R}^n) \), then
\[
- \sum_{i=1}^n \nabla_x F_{p_i^x}(x, u, Du) + \sum_{i=1}^n \sum_{j=1}^m \nabla_y F_{p_i^y}(x, u, Du) = 0 \quad \forall x \in \Omega
\]
\( k = 1, \ldots, m \)

(strong form of the Euler-Lagrange equation)
Proof:

1. We calculate

\[
\frac{\mathcal{F}(u + \varepsilon h) - \mathcal{F}(u)}{\varepsilon} = \int_{-\varepsilon}^{\varepsilon} \frac{F(x, u(x) + \varepsilon h(x), Du(x) + \varepsilon Dv(x))}{\varepsilon} \, dx
\]

the fundamental theorem of calculus

\[
= \int_{-\varepsilon}^{\varepsilon} \int_{0}^{1} \frac{d}{ds} F(x, u + s \varepsilon h, Du + s \varepsilon Dv) \, ds \, dx
\]

= \int_{-\varepsilon}^{\varepsilon} \int_{0}^{1} \left( \frac{d}{ds} \left[ F_{p_i}(x, u + s \varepsilon h, Du + s \varepsilon Dv) \right] \right) \, ds \, dx
\]

converges pointwise to

\[
F_{p_i}(x, u, Du) \frac{d}{dt} \int_{0}^{1} \left[ F_{p_i}(x, u + s \varepsilon h, Du + s \varepsilon Dv) \right] \, ds
\]

and is bounded by

\[
C \| F \|_{C^1} \| h \|_{C^1}
\]

Lebesgue's theorem of dominated convergence.

\[
(\varepsilon \to 0)
\]

\[
\int_{-\varepsilon}^{\varepsilon} \int_{0}^{1} \left( \frac{d}{ds} \left[ F_{p_i}(x, u + s \varepsilon h, Du + s \varepsilon Dv) \right] \right) \, ds \, dx
\]
So the first variation exists and
\[
\delta \mathcal{E}(u; h) = \int_\Omega \left( F_{p_i} (x, u(x), Du(x)) \delta_i h(x) \right) dx + F_{F_k} (x, u(x), Du(x)) h(x) \delta x.
\]

If now is a local minimizer, \( \hat{h} \in C^1 \), then
\[
g(x) := \mathcal{E}(u + \hat{h})
\]
has a local minimum \( u \) in \( \Omega \).
(Note that \( u + \hat{h} \) is in \( H \) as \( h \) vanishes near the boundary of \( \Omega \)).

\[
\Rightarrow \frac{d}{d\hat{h}} g(x) = \delta \mathcal{E}(u; \hat{h}) = 0,
\]
for \( \hat{h} = h \cdot e_k \), \( h \in C^1_c (\Omega, \mathbb{R}) \).

We get
\[
\int_\Omega \left( F_{p_i} (x, u(x), Du(x)) \delta_i h(x) \right) dx + F_{F_k} (x, u(x), Du(x)) h(x) \delta x
\]
\[
= 0.
\]
If \( F, u \in C^2 \) we get using partial integration

\[
\int \left( -\frac{d}{dx} F^k_p (x, u(x), Du(x)) \right) u(x) \, dx
+ \frac{d}{dx} F^k (x, u(x), Du(x)) \, u(x) \, dx
\]

Using the following Lemma, this completes the proof of the theorem.

**Lemma 4.3** (Fundamental Lemma of the Calculus of Variations)

If \( f \in C^0 (\Omega, \mathbb{R}) \) satisfies

\[
\int f \phi \, dx = 0 \quad \forall \phi \in C_c^\infty (\Omega),
\]

then

\[
f \equiv 0 \quad \forall x \in \Omega.
\]

**Proof:** Assume there is an \( x_0 \in \Omega \) with \( f(x_0) > 0 \).

Since \( f \in C^0 \) and \( \Omega \) open, there is an \( \epsilon > 0 \) such that \( B_\epsilon (x_0) \subseteq \Omega \) and \( f(x) > 0 \) for all \( x \in B_\epsilon (x_0) \).
Now we set

\[
\phi(x) = \begin{cases} 
-\frac{1}{x^2 - \varepsilon^2} & \text{if } x \in B_{\varepsilon}^c(0) \\
0 & \text{else}
\end{cases}
\]

and get

\[
\int f(x) \phi(x) \, dx = \int_{B_{\varepsilon}^c(0)} f(x) \phi(x) \, dx > 0
\]

\[
> 0 \quad \forall \varepsilon > 0
\]

Examples:

(i) **Minimal graphs**: Let \( m = 1 \) and

\[
F(x, q, p) = \sqrt{1 + |p|^2},
\]

and hence

\[
\Sigma(u) = \text{area of the graph of } u = \int \sqrt{1 + |Du(x)|^2} \, dx.
\]

Then every minimizer of \( \Sigma \) satisfies the minimal graph equation:

\[
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \quad \text{in } \mathbb{R}^2
\]