Applications: Existence of extremal functions for Sobolev's embedding

**Theorem 5.5:** There is a $u \in H^1_0 \cap H^2 (\mathbb{R})$ with

$$\|u\|_{L^{2^*}} = \|\nabla u\|_{L^2}$$

**Proof:** We construct a minimizer

$$F(u) = \int |\nabla u|^{2^*} \, dx$$

within

$$M = \{ u \in H^1_0 \cap H^2 (\mathbb{R}^n) : \int |u|^{2^*} \, dx = 1 \}$$

Let $(u_k) \subset M$ be a minimizing sequence, i.e., let

$$\lim_{k \to \infty} F(u_k) = S$$

as $k \to \infty$. 
1. **Normalization:**

We choose $R_k > 0$ and $x_k \in \mathbb{R}^n$ such that

$$
\int_{B_{R_k}^c(x_k)} |u_k|^{2^*} \, dx = \sup_{x \in \mathbb{R}^n} \int_{B_{R_k}^c(x)} |u_k|^{2^*} \, dx
$$

Exchanging $u_k$ by $\tilde{u}_k(x) := \frac{1}{R_k} u_k(R_k x)$ we get a minimizing sequence satisfying

$$
\int_{B_{R_k}^c(0)} |u_k|^{2^*} \, dx = \sup_{x \in \mathbb{R}^n} \int_{B_{R_k}^c(x)} |u_k|^{2^*} \, dx
$$

(\because \quad \| \tilde{u}_k \|_{L^{2^*}} = \| u_k \|_{L^{2^*}(\mathbb{R}^n))}.

We consider the measures

$$
\mu_k := \int |\nabla u_k|^{2^*} \, dx
$$

and

$$
\nu_k := \int |u_k|^{2^*} \, dx.
$$

We want to show that in Thm. 5.4 we have "convergence" of the $\nu_k$. 
Obviously, due to our normalization, the sequence $v_k$ does not vanish.

(\textcolor{red}{\textcircled{3}}) \textcolor{red}{\textbf{We cannot have dichotomy:}}

Let us assume that there is an $\varepsilon \in (0, 1)$ and $R_\infty \to \infty$ such that

$$\lim_{k \to \infty} \left( \int_{B_2(x_k)} d \mu_k - \varepsilon \right)$$

$$+ \left| \int_{R^N - \overline{B}_{r_0}(x_k)} \frac{d \mu_k - (1 - \varepsilon)}{d \mu_k - (1 - \varepsilon)} \right| = 0$$

We choose $\varphi \in C_0^\infty (\overline{B_2(C)})$ with $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $B_1(C)$ and set

$$\varphi_m(x) := \varphi \left( \frac{x - x_0}{R_m} \right).$$
\[ \| \nabla u_k \|_{L^2}^2 = \int \left( \left( \nabla u_k \right)^2 \psi_k + |\nabla u_k|^2 \psi_k^2 \right) dx \leq \int \left[ \left( \nabla u_k \right)^2 \psi_k^2 + \nabla u_k \left( \psi_k - \psi_k^2 \right)^2 \right] dx \]

\[ \leq \mathcal{S} \left( \int \left( \int_{B_R(x_k)} \left( \frac{1}{R} \right)^{\frac{2}{2-\alpha}} \right) \left( \int_{B_R(x_k)} \left( \frac{1}{R} \right)^{\frac{2}{2-\alpha}} \right) dx \right)^{\frac{1}{2}} \]

\[ \leq \mathcal{S} \left( \left( \frac{p}{q} + (1-\frac{p}{q}) \right)^{\frac{1}{2}} \right) \]
Applying Thm. 5.4 we get
\[ \mu = \| \nabla u \|_{L^2}^2 \, dx + \sum_{j \in J} \mu^{(j)} \, S^{(j)}(i) \, \delta_{x,i} \]
\[ v = \| u \|_{L^2}^2 \, dx + \sum_{j \in J} \varphi^{(j)}(i) \, S^{(j)}(i) \]
with
\[ S(v^{(j)}(i))^{2/\lambda} \lesssim \mu^{(j)}(i) \]

Hence
\[ S + o(1) = \| \nabla u \|_{L^2}^2 = \int_{\Omega} d|\mu| \]
\[ \geq \| \nabla u \|_{L^2}^2 + \sum_{j \in J} \mu^{(j)}(i) + o(1) \]
\[ \geq S \left( \| u \|_{L^{2^*}}^2 + \sum_{j \in J} v^{(j)}(i) \right)^{2/\lambda} + o(1) \]
\[ \geq S \left( \| u \|_{L^{2^*}}^2 + \sum_{j \in J} v^{(j)}(i) \right)^{2/\lambda} + o(1) \]
continuity \[ \frac{2}{\lambda} \]
\[ S \rightarrow S \]
\[ \| v \|_{L^2} = 1 \]

with \( \lambda = \) if and only if it must one of the terms
\[ \| u \|_{L^{2^*}} \, v(i) \left( \frac{1}{2} \right) \]
is different from 0.
\[ \Rightarrow \| u \|_{L^{2^*}} = 1 \]