Application: Existence of extremal functions for Sobolev's embedding

Theorem 5.5: There is a $u \neq 0 \in \mathring{W}^{1,2}(\mathbb{R})$ with

$$S \|u\|^2_{L^{2^*}} = \|\nabla u\|_{L^2}$$

Proof: We construct a minimizing

$$\mathcal{F}(u) = \int_{\mathbb{R}} |\nabla u|^2 dx$$

within

$$M = \{ u \in \mathring{W}^{1,2}(\mathbb{R}^n) : \int |u|^{2^*} dx = 1$$

Let $(u_k) \subset M$ be a minimizing sequence, i.e. let

$$\lim_{k \to \infty} \mathcal{F}(u_k) = S \quad \text{as } k \to \infty.$$

① Normalization:

We choose $R_k > 0$ and $x_k \in \mathbb{R}^n$ such that

$$\int_{B_{R_k}(x_k)} |u_k|^{2^*} dx = \sup_{x \in \mathbb{R}^n} \int_{B_{R_k}(x)} |u_k|^{2^*} dx$$

$$= \frac{1}{2}.$$

Exchanging u_k by $\tilde{u}_k(x) := R_k^{\frac{n-2}{n}} u_k(x_k + R_k x)$ we get a minimizing sequence satisfying

$$(*) \qquad \int_{B_1(0)} |u_k|^{2^*} dx = \sup_{x \in \mathbb{R}^n} \int_{B_1(x)} |u_k|^{2^*} dx$$

$$= \frac{1}{2}$$

(since $\|\tilde{u}_k\|_{L^{2^*}} = \|u_k\|_{L^{2^*}}$ ✓).

We consider the measures

$$\mu_k := |\nabla u_k|^2 dx$$

and $\nu_k := |u_k|^{2^*} dx$. and want to show that as in Thm. 5.4 we have "convergence" of the ν_k.

Obviously, due to our normalization the sequence v_k does not vanish

(ii) We cannot have dichotomy:

Let us assume that there is an $\lambda \in (0, 1)$ and $R_m \to \infty$ such that

$$\lim_{k \to \infty} \left(\left| \int_{B_R(x_k)} d\mu_k - \lambda \right| + \left| \int_{\mathbb{R}^n - B_{R_k}(x_k)} d\mu_k - (1-\lambda) \right| \right) = 0$$

We choose $\varphi \in C_c^\infty(B_2(0))$ with $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $B_1(0)$ and set

$$\varphi_m(x) := \varphi\left(\frac{x - x_0}{R_m} \right).$$

we get

$$\|\nabla u_k\|^2_{L^2} = \int \left(|\nabla u_k|^2 \varphi_k + |\nabla u_k|^2 \varphi_k^2 \right)$$

$$\geq \int \left(|\nabla u_k|^2 \varphi_k^2 + |\nabla u_k|^2 (1-\varphi_k)^2 \right) dx$$

$$\geq S \left(\int \left(|u_k|^{2^*} \varphi_k^{2^*} + |u_k|^{2^*} (1-\varphi_k)^{2^*} \right) dx \right)^{\frac{2}{2^*}}$$

$$\geq S \left(\int_{B_R(x_k)} |u_k|^{2^*} dx + \int_{B_{R_k}(x_k)} |u_k|^{2^*} dx \right)^{2/2^*}$$

$$\geq S \cdot \underbrace{\left(\lambda^{p/q} + (1-\lambda)^{p/q} \right)}_{>1 \text{ if } \lambda > 0} \quad ↯$$

(ii) Applying Thm. 5.4 we get

$$\mu \geq |\nabla^k u|^p \, dx + \sum_{j \in J} \mu^{(j)} \delta_{x^j}$$

$$\nu = |u|^{2^*} dx + \sum_{j \in J} \nu^{(j)} \delta_{x^j}$$

with

$$S \left(\nu^{(j)} \right)^{2/2^*} \leq \mu^{(j)}.$$

Hence,

$$S + o(1) = \|\nabla u_m\|^2_{L^2} = \int_{\mathbb{R}^m} d\mu_m$$

$$\geq \|\nabla u\|^2_{L^2} + \sum_{j \in J} \mu^{(j)} + o(1)$$

$$\geq S \left(\|u\|^{2/2^*}_{L^{2^*}} + \sum_{j \in J} \left(\nu^{(j)}\right)^{2/2^*} \right) + o(1)$$

$$\geq S \left(\|u\|_{L^{2^*}} + \sum_{j \in J} \nu^{(j)} \right)^{2/2^*} + o(1)$$

concavity of $s \to s^{2/2^*}$ & $\int d\nu = 1$

$$= S + o(1)$$

with "=" if and only if at most one of the terms

$$\|u\|_{L^{2^*}}, \quad \nu^{(j)}$$

is different from 0.

$$\Rightarrow \quad \|u\|_{L^{2^*}} = 1 \qquad \square$$