§ 6.1 A finite dimensional example

(Stampfli & Schmutzlin (1982))

Then 2.1.1. On any smooth closed surface of genus 0 there exist at least 3 prime (free of crossings) geodesics.

Model situation: (Shrink the 3-dimensional to something flat?)

\( \pi \Rightarrow \text{Ex}

We are looking for lines that come back on themselves when reflect at the boundary \( \Pi \). (We are playing a Billiard.)
Model problem:

Let $\gamma \in C^1(\mathbb{R} \setminus \mathbb{Z}, \mathbb{R}^2)$ be a parametrization of $\Gamma$ and

$$f(s, t) = \frac{1}{2} \| \gamma(s) - \gamma(t) \|^2,$$

for $s, t \in \mathbb{R}$.

$\Rightarrow f \in C^1((\mathbb{R} \setminus \mathbb{Z})^2)$ and $f(s, t) = f(t, s)$

and

$$\frac{2}{ds} f(s, t) = \dot{\gamma}(s) \cdot (\gamma(s) - \gamma(t)),$$

$$\frac{2}{dt} f(s, t) = \dot{\gamma}(t) \cdot (\gamma(s) - \gamma(t)).$$

$\Rightarrow$ The line through $\gamma(s)$ and $\gamma(t)$ is reflected onto itself at $\gamma(s)$ and $\gamma(t)$ if and only if $(s, t)$ is a critical point of $f$, (i.e., $df(s, t) = 0$).

$\square$
(i) Since \( f(s_1t) = 0 \) and \( f(s_1t) = 0 \) if \( s = t \mod \mathbb{Z} \), we get points \( s = t \mod \mathbb{Z} \) are trivial critical points (corresponding to constant curves on \( \mathbb{C} \)).

(ii) On the compact domain \( \mathbb{C}_s = (\mathbb{R}/\mathbb{Z})^2 \), \( f \) achieves its maximum at some point \( (\bar{s}, \bar{t}) \).
   ⇒ again a critical point corresponding to the diameter of \( \mathbb{S} \).
   ⇒ second prime geodesic

We try to find a "mountain path" \( \nabla \) (to get a third prime geodesic...
We try to find a saddle point \((s_1, t_1)\) of \(f\) as the point of "maximal elevation" of a "path" of "least maximal height" connecting the "valleys" \(\{(s, s) : s \in \mathbb{R}\}\) and \(\{(s, s-1) : s \in \mathbb{R}\}\).

Questions:

1. Can we find a path \(f\) of least max. height?

2. If so, does contain a critical point \((s_1, t_1)\) with

\[
f(s_1, t_1) = \inf_{p \text{ path } (s_1,t_1) \in \mathcal{P}} f(s_1, t_1)
\]

3. What happens if \(\beta_k = 0\)?
What is a path?

Let \( p_k \in C^0([0,1]) \) with \( p_k(0) = (s_0, s_0) \)
\( p_k(1) = (s_k, s_k - 1) \) and
\[
\sup_{0 < x < 1} f(p(x)) \to f_1.
\]

Problem: We cannot just deduce that
\( p_k \to p_\infty \) in \( C^0 \).

Workaround:
The sets \( \bar{\Pi} = p_k([0,1]) \subset \mathbb{R}^2 \)
are compact and connected with
\[ \bar{\Pi} \cap \left\{ (s,s) : s \in \mathbb{R} \right\} = \emptyset, \]

\[ \bar{\Pi} \cap \left\{ (s,s-1) : s \in \mathbb{R} \right\} = \emptyset. \]

(\( \bar{\Pi} \) is noted not be a curve though.)
Ad ii: Let us assume that \( df(s,t) \neq 0 \) for all \((s,t) \in \Omega \). Then we can reach a contradiction looking at

\[
\phi_\varepsilon(T)
\]

where \( \phi_\varepsilon \) is the flow in direction of \(-\nabla f\), i.e., a solution of

\[
\begin{aligned}
\frac{d}{dt} \phi_\varepsilon(x) &= -\nabla f(x) \\
\phi_0(x) &= x
\end{aligned}
\]

That is well-defined only if we assume that \( f \in C^1 \).

Example 2: Let \( f \in C^1(\mathbb{R}^2) \) with

\[
f(x,y) = e^x - y^2.
\]

Then \( M_0 = \{(x, y) : f(x, y) < 0\} \) has two components

\[
M_0^\pm = \{(x, y) : f(x, y) < 0\}
\]

\( \Rightarrow \) might expect a mountain path, but

\[
df(x, y) = (e^x, -2y) \neq 0
\]
§ 6.2 Pseudo-gradient flows.

Let $X$ be a Banach space.

Def. 6.1: Let $F \in C^1(X)$.

(i) $u \in X$ is critical for $F$ if $dF(u) = 0$, otherwise $u$ is regular.

(ii) $\beta \in \mathbb{R}$ is a critical value of $F$ if there exists a critical point $u \in X$ with $F(u) = \beta$; otherwise, $\beta$ is a regular value.

Def. 6.2 (Pseudo-gradient vector field)

Let $F \in C^1(X)$ and $\mathfrak{F} := \{ x \in X : dF(x) \neq 0 \}$.

A pseudo-gradient vector field (p.g.v.f) is a locally Lipschitz map $\widetilde{\xi} : \mathfrak{F} \to X$

with

(i) $\| \widetilde{\xi}(u) \|_X \leq 1 \quad \forall u \in \mathfrak{F}$

(ii) $\left( dF(u), \widetilde{\xi}(u) \right)_X + \frac{1}{2} \| dF(u) \|_{X^*}^2 \leq 1 \quad \forall u \in \mathfrak{F}$

$(\text{so} \quad \frac{1}{2} \leq \| \widetilde{\xi}(u) \|_X \leq 1)$. 

Theorem 6.3 (Palais, 1966)

For any \( E \in C^1(X) \) there exists \( \alpha \) (p.g. \( v \cdot f \)).

Proof:

(1) Fix \( x_0 \in X \). From the definition of \( \| d v E(u) \|_{X^*} \) we get a \( v_0 = v_0(x_0) \) with

\[
\| v_0 \|_{X^*} < 1
\]

and

\[
\langle d v E(x_0), v_0 \rangle > \frac{1}{2} \| d v E(x_0) \|_{X^*}
\]

Since \( u \rightarrow d v E(u) \) is continuous, we have

\[
\langle d v E(u), v_0 \rangle > \frac{1}{2} \| d v E(u) \|_{X^*} \quad \forall u \in U = U(x_0)
\]

\( U(x_0) \) an open neighborhood of \( x_0 \).

(2) The family \( (U(u))_{u \in X} \) is an open of \( X \), since 

\( \rightarrow \) There is a locally finite 

and \( X \) is metric 

\[
U_j \subset U(u_j) \quad \forall j \in \mathbb{I}
\]

for some

\[
U_j \subset U(u_j) \quad \forall j \in \mathbb{I}
\]

and \( \mathbb{I} \subset U \).
Let \( (\psi_j) \) for \( j \in I \) be a locally Lipschitz partition of unity subordinate to \( (U_i) \) for \( i \in I \).

(For instance, let \( \psi_j(x) = \text{dist}(x, X \setminus U_i) \) and \( C_0, \infty(\gamma) \))

\[
\psi_j(x) = \frac{\psi_j(x)}{\sum_{j \in I} \psi_j(x)}
\]

finite sum on \( U_i \). \( \Box \)

(iii) Set

\[
\tilde{z}(x) = \sum_{j \in I} \psi_j(x) v(\alpha_j), \quad x \in \mathbb{X}
\]

Then

\[
\|\tilde{z}(x)\| \leq \max_{j: \text{well}} \|v(\alpha_j)\| \leq 1
\]

and

\[
\langle d\tilde{z}(x), \tilde{z}(x) \rangle = \sum_{j \in I} \psi_j(x) \langle d\tilde{z}(x), v(\alpha_j) \rangle
\]

\[
\geq \frac{1}{2} \|d\tilde{z}(x)\|_\gamma
\]

for all \( x \in \mathbb{X} \). \( \Box \)