Remark If $\tilde{F}$ admits some group action G as symmetries, $\tilde{e}$ may be constructed to be G-equivariant.

For example: E even, i.e. $E(-u) = E(u)$

$$\rightsquigarrow \tilde{e}(u) = \frac{1}{2}\left(e(u) - e(-u)\right).$$

More generally: G compact Lie group acting on G, (i.e. there is a group homomorphism of G onto a subgroup of isometries of V s. th the evaluation map

$$G \times V \to V, (g,u) \to gu$$

is cont.) that leaves E invariant, i.e.

$$E(gu) = E(u) \quad \forall (g,u) \in G \times V$$

Setting

$$\tilde{e}(u) := \int_G g^{-1} e(gu)\, \underbrace{dg}_{\text{Haar's measure on } G}$$

we get

$$\tilde{e}(gu) = g \cdot \tilde{e}(u) \quad \forall (g,u) \in G \times V.$$

For $\beta \in \mathbb{R}$, $\delta > 0$ let

$$E_\beta := \{ u \in X : \Phi(u) < \beta \} \quad \text{(sublevel sets)}$$

$$K_\beta := \{ u \in X : d\Phi(u) = 0 ,\ E(u) = \beta \}$$

(kritical points)

$$N_{\beta,\delta} := \{ u \in X : \|dE(u)\|_{X^*} \leq \delta ,\ |E(u) - \beta| < \beta \}$$

<u>Lemma 6.4:</u> Let $\mathcal{F} \in C^1(X)$ and $\beta \in \mathbb{R}$, $\hat{\varepsilon} > 0$ and suppose that $N_{\beta,\delta} \neq \emptyset$. Then there is an $0 < \varepsilon < \hat{\varepsilon}$ and $\phi \in C^0(X \times [0,1], X)$ such that

i) $\phi(\cdot, t) : X \to X$ is a homeomorph. $\forall t \in [0,1]$

ii) $\Phi(u,t) = u$ if $t = 0$ or $\mathcal{F}'(u) = 0$ or $|\mathcal{F}(u) - \beta| \geq \hat{\varepsilon}$

iii) $t \to E(\phi(u,t))$ is non-increasing

iv) $\Phi(\mathcal{F}_{\beta+\varepsilon}, 1) \subset \mathcal{F}_{\beta-\varepsilon}$

Proof: (i) Let $\varepsilon = \frac{1}{4} \min\{\tilde{\varepsilon}, \delta\} > 0$.
Choose a smooth function $0 \le \tau \le 1$ such that

$$\tau(s) = \begin{cases} 1, & \text{if } |s-\beta| \le \varepsilon \\ 0, & \text{if } |s-\beta| \ge 2\varepsilon. \end{cases}$$

Since $N_{\beta,\delta} = \emptyset$, we get that all $u \in X$ with $d\mathcal{E}'(u) = 0$ satisfy

$$|E(u) - \beta| \ge \delta \ge 4\varepsilon > 2\varepsilon$$

$\leadsto$ $\tau(E(v))$ vanishes near critical points

$\leadsto$

$$e(u) := \begin{cases} \tau(E(u))\,\tilde{e}(u), & u \in \tilde{X} \\ 0 & \text{else} \end{cases}$$

is a well defined locally Lipschitz map.

(ii) Let $\phi : X \times \mathbb{R} \to X$ be the flow induced by $-e$, that is

$$\partial/\partial t \; \phi(u,t) = -e(\Phi(u,t))$$

$$\Phi(u, 0) = u$$

$\leadsto$ (i) & (ii) hold by construction.
Furthermore,

$$\frac{d}{dt} \mathcal{E}(\phi(u,t)) = \langle \mathcal{E}'(\phi(u,t)), \frac{\partial\Phi}{\partial t}(u,t) \rangle$$

$$= -\langle \mathcal{E}'(\phi(u,t)), \underbrace{e(\phi(u,t))}_{= \tau(\phi(u,t))\, \tilde{e}(\phi(u,t))} \rangle$$

$$= - \underbrace{\tau(\tilde{\mathcal{F}}(\phi(u,t))}_{\geq 0} \cdot \underbrace{\langle \tilde{\mathcal{F}}'(\phi(u,t)), e(\phi(u,t))\rangle}_{> \frac{1}{2}\delta}$$

$= 1$ if $|\tilde{\mathcal{F}}(\phi(u,t))| - \beta| < \varepsilon < \delta$

$|\tilde{\mathcal{F}}(u) - \beta| < \varepsilon$

$\leq -2\varepsilon$;

$\rightsquigarrow$ (iii) & (iv) follow $\nabla_{\!\circ}$

//

Near a point of "mountain pass" we cannot apply the last Lemma $\nabla_{\!\circ}$

Lemma 6.5: Let $\mathcal{F} \in C^1(X)$ and suppose that

$$\tilde{\mathcal{F}}_0 := \{ u \in X : \mathcal{F}(u) < 0 \}$$

has at least two components.
For $u_1, u_0 \in \tilde{\mathcal{F}}_0$ in two different components we let

$$\beta = \inf_{\gamma \in \Gamma} \sup_{0 \leq s \leq 1} \mathcal{F}(\gamma(s)) \geq 0$$

where

$$\Gamma = \{ \gamma \in C^0([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1 \}.$$

Then $N_{\beta,\delta} \neq \emptyset$ for all $\delta > 0$.

//

Proof: (By contradiction)

Suppose $\mathcal{N}_{\beta,\delta} = \emptyset$ for an $\delta > 0$.

Lemma 1

with $\bar{\varepsilon} = \beta - \max\{\tilde{\mathcal{F}}(u_0), \tilde{\mathcal{F}}(u_1)\} > 0$ $\xrightarrow{}$ get $\varepsilon \in [0, \bar{\varepsilon}]$

and $\Phi : X \times [0,1] \to X$ with all the properties.

Def of β

$\rightsquigarrow$ $\exists\ \gamma \in \Gamma$ with

$$\sup_{0 \le s \le 1} \tilde{\mathcal{F}}(\gamma(s)) < \beta + \varepsilon$$

$\rightsquigarrow$ $\gamma_1(s) := \Phi(\gamma(s), 1)$ satisfies

$$\gamma_1(u_0) = u_0 \qquad \gamma_1(u_1) = u_1 \quad \text{by (ii)}$$

and by (iv)

$$\sup_{0 \le s \le 1} \tilde{\mathcal{F}}(\gamma_1(s)) < \beta - \varepsilon \qquad \lightning$$

//

So in the "mountain pass" setting we get

$$\mathcal{N}_{\beta,\delta} \neq \emptyset \qquad \forall \delta > 0$$

Does this imply $K_\beta \neq \emptyset$? No !
(c.f. Example 2 !)

<u>But:</u> If X were compact, $\forall$ for $u_k \in \mathcal{N}_{\beta, 1/k}$ a subsequence we would have $u_k \to u$

$$\tilde{\mathcal{F}}(u) = \lim_{k\to\infty} \tilde{\mathcal{F}}(u_k)$$

and
$$\|\tilde{\mathcal{F}}'(u)\|_{X'} = \lim_{k\to\infty} \|\tilde{\mathcal{F}}'(u_k)\|_{X'} = 0$$

This consideration leads to

<u>Definition</u> 6.6 (Palais-Smale condition)

(i) $(u_k) \subset X$ is a <u>Palais-Smale</u> sequence at level β ($(P.-S)_\beta$-sequence) if

$$\tilde{\mathcal{F}}(u_k) \xrightarrow{k\to\infty} \beta, \quad \|\tilde{\mathcal{F}}'(u_k)\|_{X'} \xrightarrow{k\to\infty} 0$$

(ii) $\tilde{\mathcal{F}}$ satisfies $(P.-S.)_\beta$ if any $(P.S.)_\beta$-sequence has a convergent subsequence.

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Examples:

(i) $\mathcal{F}(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx$ on $X = W_0^{1,2}(\Omega)$

satisfies $(P.S)_\beta$ for every $\beta \in \mathbb{R}$

Proof:

$(u_k) \subset W_0^{1,2}(\Omega)$ $(P.S)_\beta$-sequence

i.e

$$\int |\nabla u_k|^2 dx \to 2\beta$$

$$-\Delta u_k \to 0 \text{ in } \left(W_0^{1,2}(\Omega)\right)'$$

after going to a subsequence and

$$u_k \xrightarrow{\omega} u \text{ in } W_0^{1,2}(\Omega)$$

$$o(1) = \langle -\Delta u_k, u_k - u \rangle = \int_\Omega \nabla(u_k - u) \cdot \nabla(u_k - u)\, dx + \int_\Omega \nabla u \, \underbrace{\nabla(u_k - u)}_{\to 0 \text{ weakly}}\, dx$$

$$= \|\nabla(u_k - u)\|^2_{L^2(\Omega)} + o(1)$$

with $o(1) \to 0$ as $k \to \infty$.

$\Rightarrow$ $u_k \to u$ in $W_0^{1,2}(\Omega)$ //

(ii) Let $\Omega \subset\subset \mathbb{R}^n$, $2 < p < 2^* = \frac{2n}{n-2}$ (if $n > 2$)

$$\mathcal{F}(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{p} \int_\Omega |u|^p dx,$$

for $u \in W_0^{1,2}(\Omega)$.

Claim: $\mathcal{F}$ satisfies $(P.-S.)_\beta$ at every $\beta \in \mathbb{R}$ //

Proof: (i) Let $(u_k) \subset W_0^{1,2}(\Omega)$ be a $(P.-S.)_\beta$ - sequence. Then

$$o(1) \cdot \|u_k\|_{W_0^{1,2}} = \langle \mathcal{F}'(u_k), u_k \rangle = \int_\Omega \left(|\nabla u_k|^2 - |u_k|^p \right) dx$$

and hence

$$p \cdot \beta + o(1) \cdot \|u_k\|_{W_0^{1,2}} = p\, \mathcal{F}(u_k) - \langle \mathcal{F}'(u_k), u_k \rangle$$

$$= \frac{p-2}{2} \int_\Omega |\nabla u_k|^2 dx$$

$$\geq c_0 \, \|u_k\|_{W_0^{1,2}(\Omega)}$$

$\rightsquigarrow$ $(u_k) \subset W_0^{1,2}(\Omega)$ is bounded.

subsequence $\rightsquigarrow$ $\overset{\nabla}{\circ}$ Can assume $u_k \overset{w}{\rightarrow} u$ in $W_0^{1,2}$ and $u_k \rightarrow u$ in L^p

(ii) $$o(1) = \langle \tilde{S}'(u_k), u_k - u \rangle$$

$$= \int \Big(\nabla u_k \nabla (u_k - u) - \underbrace{u_k |u_k|^{p-2}}_{\|\cdot\|_{L^{p/p-1}} \leq C} \underset{\substack{\downarrow L^p \\ 0}}{(u_k - u)} \Big) dx$$

$$= \int \nabla(u_k - u) \nabla(u_k - u) \, dx + o(1)$$

$$= \| \nabla (u_k - u) \|^2_{L^2} + o(1)$$

$\square$

We can now proof.

<u>Theorem 6.7:</u> Let $\tilde{\mathcal{F}} \subset C^1(X)$ and suppose that

$$\tilde{\mathcal{F}}_0 = \{ u \in X : \tilde{\mathcal{F}}(u) < 0 \}$$

has at least two components. Let u_0, u_1 belong to different components of $\tilde{\mathcal{F}}_0$ and define

$$\beta = \inf_{\gamma \in \Gamma} \sup_{0 \le s \le 1} \tilde{\mathcal{F}}(\gamma(s)) \ge 0$$

where

$$\Gamma = \{ \gamma \in C^1([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1 \}$$

Assume that $\tilde{\mathcal{F}}$ satisfies $(P.-S.)_\beta$.
Then $K_\beta \neq 0$. //

<u>Proof:</u> By Lemma 6.

$$\mathcal{N}_{\beta,\delta} := \{ u \in X : |\tilde{\mathcal{F}}(u) - \beta| < \delta, \| \tilde{\mathcal{F}}'(u) \|_{X'} < \delta \}$$

$$\neq \emptyset \quad \forall \delta > 0.$$

Pick $u_k \in \mathcal{N}_{\beta, 1/k}$. Since (u_k) is a $(P-S)_\beta$ - sequence, we get a subsequence converging to $u \in K_\beta$ //

Corollary 6.8 (THE "mountain-pass lemma"),

Suppose $\tilde{\mathcal{F}} \in C^1(X)$ sat. $(P-S)_\beta$ for every $\beta \in \mathbb{R}$. Assume

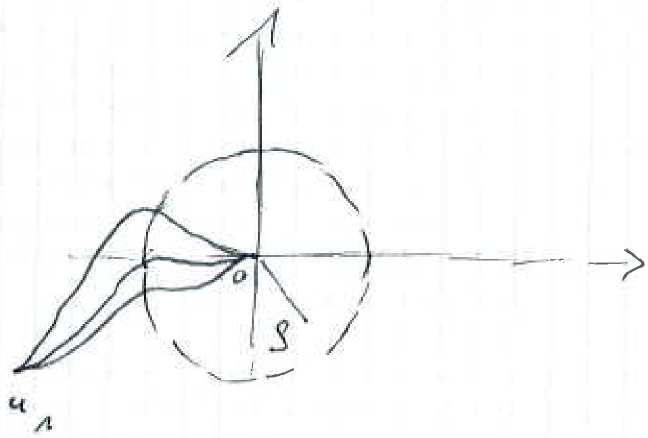
(i) $\tilde{\mathcal{F}}(0) = 0$

(ii) $\exists\, \rho, \alpha > 0$: $\|u\|_X = \rho \Rightarrow \tilde{\mathcal{F}}(u) \geq \alpha$

(iii) $\exists\, u_1 \in X$: $\|u_1\| > \rho$, $\tilde{\mathcal{F}}(u_1) \leq 0$.

Then $K_\beta \neq \emptyset$ for some $\beta \geq \alpha$

//



Proof: $\tilde{\mathcal{F}}_\alpha$ has different components, one containing $u_0 = 0$ and one containing u_1. Apply Theorem 6. (to $\tilde{\mathcal{F}} = \tilde{\mathcal{F}} - \alpha$)

//

Example: $\tilde{\mathcal{F}}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx$,

$u \in W_0^{1,2}(\Omega)$, $2 < p < 2^*$, $\Omega \subset\subset \mathbb{R}^n$

satisfies $(P-S)_\beta$ for every $\beta \in \mathbb{R}$ and $\tilde{\mathcal{F}}(0) = 0$.

$$\tilde{f}(u) = \frac{1}{2}\|\nabla u\|^2_{L^2} - \frac{1}{p}\|u\|^p_{L^p}$$

$$\geq c_0 \|u\|^2_{W^{1,2}_0(\Omega)} - c_1 \|u\|^p_{W^{1,2}_0(\Omega)}$$

$$\underset{\rho = \|u\|_{W^{1,2}_0(\Omega)}}{=} c_0 \|u\|^2_{W^{1,2}_0(\Omega)} \underbrace{\left(1 - \frac{c_1 \rho^{p-2}}{c_0}\right)}_{1/2}$$

$$= c_0 \frac{1}{2} \|u\|^2_{W^{1,2}_0(\Omega)} \qquad \text{if } \rho = \left(\frac{c_0}{2c_1}\right)^{\frac{1}{p-2}}$$

and

$$\tilde{f}(\lambda u) = \frac{\lambda^2}{2}\|\nabla u\|^2_{L^2} - \frac{\lambda^p}{p}\|u\|^p_{L^p}$$

$$\to -\infty \quad \text{if } \lambda \to \infty$$

for all $u \not\equiv 0$.